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Stochastic Delay Equations and Inclusions with Mean Derivatives on Riemannian Manifolds

Yuri E. Gliklikh[a] **and Salah E.A. Mohammed**[b]

[a]Mathematics Faculty
Voronezh State University
394006 Voronezh, Russia
yeg@math.vsu.ru

[b]Department of Mathematics
Southern Illinois University
Carbondale IL 62901
salah@sfde.math.siu.edu

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Abstract

We establish the existence of solutions to stochastic delay equations and inclusions with mean derivatives on a stochastically complete Riemannian manifold. The delays in both the equations and the inclusions are expressed in terms of stochastic Riemannian parallel translation.

Key words: Stochastically complete Riemannian manifold; Riemannian parallel translation; mean derivative; quadratic mean derivative; equation with delay; inclusion with delay.

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Introduction

The concept of mean derivative was introduced by Edward Nelson in his study of stochastic mechanics, a version of quantum mechanics. The equations of motion in Nelson's theory (called the Newton-Nelson equations) were the first examples of equations with mean derivatives. Later on such differential equations arose also in analyzing the motion of a viscous incompressible fluid and in the study of Navier-Stokes vortices, etc.

In all of the above-mentioned cases, the solutions of the equations are assumed to be Itô diffusion-type processes whose diffusion terms are given a priori. This is because, roughly speaking, Nelson's mean derivatives yield only the drift term (forward, backward, etc.) of a stochastic process. As an extension of Nelson's mean derivative, a new kind of *quadratic mean derivative* is introduced in [1], which encodes the diffusion term of the random process. Knowledge of the mean and quadratic mean derivatives has made it possible to recover the full diffusion process. The recent monograph by the first author [4] contains the beginnings of a general theory of equations and inclusions with mean and quadratic mean derivatives.

The present paper is devoted to a new aspect of the theory of stochastic equations with mean derivatives. More specifically, we study stochastic delay differential equations and inclusions with mean derivatives on Riemannian manifolds. In these dynamical systems the delayed terms are represented through stochastic parallel translation along random paths on the Riemannian manifold.

Our study of such systems is motivated by the following physical considerations. Dynamical systems studied in this article relate to the description of mechanical motion on non-linear configuration spaces. Within this setting, Radon (see [5]) realized that Riemannian parallel translation along a curve on the manifold is a natural representation of the notion of constancy on the entire flat configuration space. In particular, the delayed term with parallel translation describes the natural delay on curved configuration spaces. Furthermore, for controlled dynamical systems, the set-valued right hand side of an inclusion represents all possible values of the controlling parameters at the given point. The use of mean derivatives makes it possible to deal with many of the above-mentioned problems in mathematical physics.

A class of well-posed stochastic functional differential equations (sfde's) on Riemannian manifolds has been studied in [6]. The delay term in this class of sfde's is expressed in terms of Riemannian parallel translation.

For the necessary preliminary concepts and facts in this article, the reader may refer to [4].

1 Mean derivatives on a manifold

We begin this section by describing the concepts of mean and quadratic mean derivatives for a random process on flat space $\mathbb{R}^n$. Using the intrinsic nature of the mean derivatives, we are then able to lift our definitions to random processes on a smooth manifold with a connection.

Consider a stochastic process $\xi : [0,T] \times \Omega \to \mathbb{R}^n$, on a certain probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and such that $\xi(t) \in L^1(\Omega, \mathbb{R}^n$ is an L_1 random variable for each $t \in [0,T]$. Such a process determines a family of three sub- σ-algebras of the σ-algebra $\mathcal{F}$ ([4]):

(i) "the past" $\mathcal{P}_t^\xi$ generated by inverse images of Borel sets from $\mathbb{R}^n$ under all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $0 \le s \le t$;

(ii) "the future" $\mathcal{F}_t^\xi$ generated by inverse images of Borel sets from $\mathbb{R}^n$ under all mappings $\xi(s) : \Omega \to \mathbb{R}^n$ for $t \le s \le T$;

(iii) "the present" ("now") $\mathcal{N}_t^\xi$ generated by inverse images of Borel sets from $\mathbb{R}^n$ under the mapping $\xi(t) : \Omega \to \mathbb{R}^n$.

All the above σ-algebras are assumed to be complete by including all P-null sets.

As usual, we denote the conditional expectation with respect to a sub-σ-algebra $\mathcal{B}$ of $\mathcal{F}$ by the symbol $E(\cdot|\mathcal{B})$.

Following E. Nelson we adopt the following definition of forward mean derivative.

Definition 1. *The* **forward mean derivative** $D\xi(t)$ *at time* $t \in [0,T]$ *of a random process* $\xi : [0,T] \times \Omega \to \mathbb{R}^n$ *is defined by*

$$D\xi(t) := \lim_{\triangle t \to +0} E\left(\frac{\xi(t+\triangle t) - \xi(t)}{\triangle t} \middle| \mathcal{P}_t^\xi \right), \tag{1.1}$$

where the limit is assumed to exist in $L_1(\Omega, \mathcal{F}, \mathsf{P}; \mathbb{R}^n)$ *and* $\triangle t \to +0$ *stands for* $\triangle t$ *tends to* 0 *and* $\triangle t > 0$.

In the spirit of Nelson's mean derivative, the first author introduced the following concept of *quadratic mean derivative* $D_2\xi$ for a random process $\xi : [0,T] \times \Omega \to \mathbb{R}^n$ ([1]):

Definition 2. *The* **quadratic mean derivative** $D_2\xi(t)$ *at time* $t \in [0,T]$ *of a random process* $\xi : [0,T] \times \Omega \to \mathbb{R}^n$ *is defined by the relation*

$$D_2\xi(t) := \lim_{\triangle t \to +0} E\left(\frac{(\xi(t+\triangle t) - \xi(t))(\xi(t+\triangle t) - \xi(t))^*}{\triangle t} \middle| \mathcal{N}_t^\xi \right), \tag{1.2}$$

In the above expression, the increment $(\xi(t+\triangle t) - \xi(t))$ *is considered as a column vector in* $\mathbb{R}^n$*, and its transpose* $(\xi(t+\triangle t) - \xi(t))^*$ *a row vector; the limit is assumed to exist in* $L_1(\Omega, \mathcal{F}, \mathsf{P}; \mathbb{R}^{n\times n})$.

We emphasize that, in spite of the fact that the above matrix multiplication of the column and row vectors yields a matrix of rank 1, after passing to limit and taking conditional expectation, the quadratic mean derivative $D_2\xi : [0,T] \times \Omega \to \mathbb{R}^{n\times n}$ (when it exists) becomes a symmetric-nonnegative-definite-matrix-valued random process (and in many cases positive-definte).

Note that the defining relation in Definition 2 can also be written as

$$D_2\xi(t) = \lim_{\triangle t \to +0} E\left(\frac{(\xi(t+\triangle t) - \xi(t)) \otimes (\xi(t+\triangle t) - \xi(t))}{\triangle t} \middle| \mathcal{N}_t^{\xi} \right). \tag{1.3}$$

Now let M be a smooth n-dimensional manifold with a given smooth connection on its tangent bundle $TM \to M$. Suppose $\xi : [0,T] \times \Omega \to M$ is a semimartingale on M. Let m be a point on M and consider a normal chart $\phi_m : U_m \subset M \to \phi_m(U_m) \subset T_mM$ at m with respect to the above connection. Denote by τ_m the Markov time of first hitting the boundary of U_m by $\xi(t)$. Consider the measurable map $L_m : [0,T] \times U_m \to T_mM$ (see, e.g., [8])

$$L_m(t, m') := \lim_{\triangle t \to 0} E\left(\frac{\phi_m(\xi((t+\triangle t) \wedge \tau_m)) - \phi_m(\xi(t))}{\triangle t} \middle| \xi(t) = m' \right) \in T_mM$$

for any $m' \in U_m$, where $(t + \triangle t) \wedge \tau_m = min((t + \triangle t), \tau_m)$. Then there is a globally-defined measurable vector field $Y^0 : [0,T] \times M \to TM$ such that

$$Y^0(t, m) = L_m(t, m) \in T_mM$$

for every $m \in M$.

Definition 3. *For any $t \in [0,T]$, $D\xi(t) := Y^0(t, \xi(t)) \in T_{\xi(t)}M$ is called the* **mean forward derivative** *of the process $\xi : [0,T] \times \Omega \to M$ at time t (cf. [7]).*

The quadratic mean derivative $D_2\xi$ of a process $\xi : [0,T] \times \Omega \to M$ on a smooth n-dimensional manifold M can be defined in a similar manner to the mean forward derivative by using the corresponding localization of formula (3). However a connection on the manifold is not needed in this case. The quadratic mean derivative is a measurable symmetric $(2,0)$-tensor-valued random field $D_2\xi : [0,T] \times \Omega \to TM \otimes TM$; that is $D_2\xi(t) \in T_{\xi(t)}M \otimes T_{\xi(t)}M$ a.s. for all $t \in [0,T]$ (see [4]).

For the rest of this paper, we consider processes on a Riemannian manifold M and their mean derivatives with respect to the canonical Levi-Civita connection on the tangent bundle $TM \to M$.

2 The main results

Assume that M is a smooth n-dimensional Riemannian manifold with the induced Levi-Civita connection.

In principle, one can recover a semimartingale $\xi : [0,T] \times \Omega \to M$ on M from its mean derivatives $D\xi$ and $D_2\xi$. The forward mean derivative $D\xi$ gives information about the drift while the quadratic mean derivative $D_2\xi$ yields information about the diffusion term of the process. For more details the reader may consult [4].

Following [4] we denote by $\Gamma_{t,s} : T_{\xi(s)}M \to T_{\xi(t)}M$ the operator of parallel translation along a semimartingale $\xi : [0,T] \times \Omega \to M$ under the Levi-Civita connection.

Consider two vector fields $X, Y : [0,T] \times M \to TM$ on M and let $h > 0$ be a given fixed delay. In this section we will study solutions of the following *stochastic delay equation with mean derivatives*:

$$\begin{aligned} D\xi(t) &= X(t,\xi(t)) + \Gamma_{t,t-h}Y(t,\xi(t-h)), \\ D_2\xi(t) &= I, \quad t \in [0,T] \\ \xi(t) &= \phi(t), \quad t \in [-h,0]. \end{aligned} \tag{2.1}$$

In the above stochastic delay equation, I is the identity matrix and the initial condition $\phi : [-h,0] \to M$ is a smooth curve on M.

Definition 4. *The stochastic delay equation* (2.1) *is said to have a solution on an interval* $[-h,l)$ *if there exist a probability space* $(\Omega, \mathcal{F}, \mathsf{P})$ *and a semimartingale process* $\xi : [-h,l) \times \Omega \to M$*, such that for* $t \in [-h,l)$ *equation* (2.1) *is satisfied.*

Fix a point $m_0 \in M$ and $T > 0$. Consider the Banach space $C^0([0,T], T_{m_0}M)$ of continuous curves $[0,T] \to T_{m_0}M$ with the usual supremum norm. Let $\mathcal{C}$ denote the σ-algebra generated by cylinder sets in $C^0([0,T], T_{m_0}M)$. For each $t \in [0,T]$, denote by $\mathcal{C}_t$ the sub-σ-algebra of $\mathcal{C}$ generated by cylinder sets based on $[0,t]$.

In the discussion below, take $m_0 = \varphi(0)$.

Theorem 1. *Assume that for a given* $T > 0$ *the vector fields* $X, Y : [0,T] \times M \to TM$*, are jointly Borel measurable and uniformly bounded and the Riemannian manifold* M *is stochastically complete. Then for every initial* C^1 *path* φ *there exists a solution* $\xi : [-h,T] \times \Omega \to M$ *of the stochastic delay equation* (2.1)*.*

Proof. To prove global existence of a solution to the stochastic delay equation (4), we use forward steps of length $h > 0$.

To begin with, suppose that $0 < T \leq h$. Consider the standard Wiener process $\tilde{w}$ on $T_{m_0}M$, i.e., the coordinate process $\tilde{w}\big(t, x(\cdot)\big) = x(t)$ on the probability space $(C^0\big([0,T], T_{m_0}M\big), \mathcal{C}, \nu)$, where ν is the Wiener measure. Recall that a sample point in $C^0\big([0,T], T_{m_0}M\big)$ is a continuous curve $x(\cdot) \in C^0\big([0,T], T_{m_0}M\big)$. Observe that $\tilde{w}$ is adapted to the filtration $\{\mathcal{C}_t\}_{0 \leq t \leq T}$.

Since M is stochastically complete, the Eells-Elworthy development $R_{EE}\tilde{w}(\cdot)$, i.e., the Wiener process on M, see [2], is well-defined for all $t > 0$. Besides Eells-Elworthy development R_{EE}, we shall use also the so-called Itô development R_I (see, e.g., [4]) whose construction is quite analogous to R_{EE} but involves equations of Itô type instead of the ones of Stratonovich type. For the Wiener process $\tilde{w}$ both developments coincide and so $R_I\tilde{w}(\cdot)$ is well-defined for all $t > 0$ (see [4]).

It is shown in [4] that $R_Ix(t)$ and the parallel translation along $x(\cdot)$ exist for ν-almost all $x(\cdot) \in C^0\big([0,T], T_{m_0}M\big)$. Furthermore, using the properties of parallel translation and of the development R_I, it follows that the stochastic process $\Gamma_{0,t}X\big(t, R_I\tilde{w}(t)\big)$, $0 \le t \le T$, in $T_{m_0}M$ is uniformly bounded and adapted to the filtration $\{\mathcal{C}_t\}_{0\le t\le T}$. Since $\varphi(\cdot)$ is at least C^1-smooth, the ordinary parallel translation is well-defined along it and we can construct the curve $\Gamma_{0,t-h}Y(t,\varphi(t-h))$, $t \in [0,T]$, in $T_{m_0}M$. Note that by the properties of parallel translation this curve is uniformly bounded. Define the process $a(t) := \Gamma_{0,t}X\big(t, R_I\tilde{w}(t)\big) + \Gamma_{0,t-h}Y(t,\varphi(t-h))$, $t \in [0,T]$, in $T_{m_0}M$. Consider the measure μ on $(C^0\big([0,T], T_{m_0}M\big), \mathcal{C})$ with density ρ with respect to ν given by

$$\rho\,(x(\cdot)) = \exp\left(\int_0^T \langle a(t), d\tilde{w}(t)\rangle - \frac{1}{2}\int_0^T a(t)^2\, dt\right). \qquad (2.2)$$

It is known (see [3]) that under the hypotheses of the theorem

$$\int_{C^0\big([0,l], T_{m_0}M\big)} \rho\, d\nu = 1, \qquad (2.3)$$

i.e., μ is a probability measure, and, furthermore,

$$w\big(t, x(\cdot)\big) = x(t) - \int_0^t a(\tau)\, d\tau$$

is a Wiener process on $(C^0\big([0,T], T_{m_0}M\big), \mathcal{C}, \mu)$ relative to $\{\mathcal{C}_t\}_{0\le t\le T}$ (see [3]). It is not hard to show that $\rho > 0$ everywhere, i.e., ν is absolutely continuous with respect to μ and has density ρ^{-1}. In other words, the probability measures μ and ν are equivalent. The coordinate process $z\big(t, x(\cdot)\big) = x(t), 0 \le t \le T$ on $(C^0\big([0,T], T_{m_0}M\big), \mathcal{C}, \mu)$ is adapted to the filtration $\{\mathcal{C}_t\}_{0\le t\le T}$. Furthermore, $\mathcal{C}_t = \mathcal{P}_t^z$ for all $t \in [0,T]$. Since the measures μ and ν are equivalent, $R_I(z(t, x(\cdot)) = R_Ix(t)$ exists μ-a.s. Thus, $z(t)$ and $w(t)$ are related via the equation

$$dz(t) = a(t)\, dt + dw(t), \quad 0 \le t \le T \qquad (2.4)$$

on $T_{m_0}M$. By definition, the process $R_Iz(\cdot)$ exists on $(C^0\big([0,T], T_{m_0}M\big), \mathcal{C}, \mu)$. Introduce the process $\xi(t)$, $t \in [-h, T]$, that coincides with $R_Iz(t)$ for $t \in [0,T]$ and with $\varphi(t)$ for $t \in [-h, 0]$. It is shown in [4] that $D\xi(t) = \Gamma_{t,0}a(t)$. But by construction $\Gamma_{t,0}a(t) = X(t, \xi(t)) + \Gamma_{t,t-h}Y(t, \xi(t-h))$. The fact that $D_2\xi(t) = I$ for $t \in [0,T]$ follows from [4]. Thus, $\xi(t), t \in [0,T]$, is a solution of (2.1) in the case $0 < T \le h$ that we are considering.

The next step is the case where $h \le T$. We will treat the case $h \le T \le 2h$. For arbitrary $T > h$ the result will follow by induction. We argue in a similar way to the above but with the following modification. For $t \in [0, h]$ we define the process $a(t)$ by the same formula $a(t) := \Gamma_{0,t} X\big(t, R_I \tilde{w}(t)\big) + \Gamma_{0,t-h} Y(t, \varphi(t-h))$; but for $t \in [h, T]$ by the formula $a(t) := \Gamma_{0,t} X\big(t, R_I \tilde{w}(t)\big) + \Gamma_{0,t-h} Y(t, R_I \tilde{w}(t-h))$ where the parallel translation along the stochastic part of the process is employed. □

Now let us turn to the case of inclusions. In this case we deal with set-valued vector fields $\mathbf{X}(t, \cdot)$ and $\mathbf{Y}(t, \cdot)$, i.e. in the tangent space $T_m M$ of every point $m \in M$ certain closed sets $\mathbf{X}(t, m)$ and $\mathbf{Y}(t, m)$ depending on time t, are given. We suppose that those set-valued vector fields are Borel measurable in the sense of set-valued maps. Recall in particular, that such vector fields have Borel measurable selectors.

Consider the system

$$
\begin{aligned}
D\xi(t) \in \mathbf{X}(t, \xi(t)) + \Gamma_{t,t-h} \mathbf{Y}(t, \xi(t-h)),\\
D_2 \xi(t) = I,\\
\xi(t) = \phi(t), \quad t \in [-h, 0].
\end{aligned}
\tag{2.5}
$$

that is a simple example of stochastic differential inclusions in terms of mean derivatives with delay on Riemannian manifolds, analogous to (2.1). The notion of solution of (2.5) is quite analogous to that of (2.1)

Theorem 2. *Assume that for a certain $T > 0$ the set-valued vector fields $\mathbf{X}(t, m)$ and $\mathbf{Y}(t, m)$ are Borel measurable and uniformly bounded for $t \in [0, T]$. Suppose also that the Riemannian manifold M is stochastically complete. Then for every initial C^1 path φ there exists a solution $\xi(t)$ of* (2.5) *for all $t \in [-h, T]$.*

Theorem 2 is reduced to Theorem 1 in the following way. As mentioned above, the Borel measurable set-valued vector fields have Borel measurable selectors. Take such selectors $X(t, m)$ and $Y(t, m)$ of $\mathbf{X}(t, m)$ and $\mathbf{Y}(t, m)$, respectively. From the hypothesis it follows that $X(t, m)$ and $Y(t, m)$ satisfy the conditions of Theorem 2. Then by Theorem 2 equation (2.1) with those $X(t, m)$ and $Y(t, m)$ has a solution that is also a solution of (2.5).

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