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Some Properties of Force Fields on the Groups of Diffeomorphisms of the flat n-Dimensional Torus, connected with the Notion of Parallelism

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Abstract

Some existence of solution theorems are proved for second order differential inclusions on the groups of diffeomorphisms of a flat n-dimensional torus. The technical tool for the proofs is the use of the notion of parallelism on those groups.

Key words: Flat torus; groups of diffeomorphisms; differential inclusions; parallelism

1 Introduction and preliminaries

We investigate second order differential equations and inclusions on the group $D^s(\mathcal{T}^n)$ of Sobolev H^s-duffeomorphisms of flat n-dimensional torus $\mathcal{T}^n$, $s > \frac{n}{2} + 1$. The necessary preliminaries on their group and Hilbert manifold structures can be found in [2, 4].

Besides the group structure mentioned above, on $D^s(\mathcal{T}^n)$ there exists an additional structure generated by the global parallelism of the tangent bundle on $\mathcal{T}^n$. This structure is the main technical tool of our consideration. It is described as follows (see, e.g., [4]).

Definition 1. *Introduce the operators:*

(i) $B : T\mathcal{T}^n \to R^n$, *the projection to the second factor in* $T\mathcal{T}^n = \mathcal{T}^n \times R^n$*;*

(ii) $A(m) : R^n \to T_m\mathcal{T}^n$, *the inverse to* B *(see (i)) mapping to the tangent space to* $\mathcal{T}^n$ *at* $m \in \mathcal{T}^n$*;*

(iii) $Q_g = A(g(m)) \circ B$, *a linear isomorphism* $Q_g : T_m\mathcal{T}^n \to T_{g(m)}\mathcal{T}^n$, *where* $g \in D^s$ *and* $m \in \mathcal{T}^n$ *.*

The operator Q_e is a one, different from the right shift, that sends every tangent space to the group isomorphically to the tangent space at the unit e. Thus, besides the right-invariant vector fields on $D^s(\mathcal{T}^n)$ there is another class of fields with a property of invariance, this time with respect to the action of operator Q. We call these fields parallel.

Definition 2. *A vector filed* X *on* $D^s(\mathcal{T}^n)$ *is called parallel if at every point* $\eta \in D^s(\mathcal{T}^n)$ *its value* $X_\eta = Q_\eta X_e$ *where* $X_e \in T_e D^s(\mathcal{T}^n)$*.*

Note that the parallel vector field X is invariant with respect to Q_θ for every $\theta \in D^s(\mathcal{T}^n)$.

By i we denote an isometric embedding of $\mathcal{T}^n$ to a Euclidean space $\mathbb{R}^k$ for k large enough, that exists by well-known Nash's theorem.

This embedding $i : \mathcal{T}^n \to \mathbb{R}^k$ generates the embedding of $D^s(\mathcal{T}^n)$ to the Hilbert space $H^s(\mathcal{T}^n, \mathbb{R}^k)$, which we denote by the same symbol i.

Recall that a tubular neighborhood U of the submanifold $iD^s(\mathcal{T}^n)$ in $H^s(\mathcal{T}^n, \mathbb{R}^k)$ has the structure of direct product $U = iD^s(\mathcal{T}^n) \times W$, where W is a ball in the space normal to the tangent space $T_e D^s(\mathcal{T}^n)$, $e = id$ is the unit in the group $D^s(\mathcal{T}^n)$. By r we denote the retraction $r : U \to D^s(\mathcal{T}^n)$. Thus the tangent spaces to U are represented as $T_\xi U = T_{r\xi} D^s(\mathcal{T}^n) \times T_\xi W$. If $X(\eta)$ is a vector field on $D^s(\mathcal{T}^n)$, the tangent map Ti sends it into the vector field TiX on $iD^s(\mathcal{T}^n)$. By symbol $\bar{X}$ we denote the extension of TiX to U of the form $\bar{X}_\xi = (TiX_{r\xi}, 0)$.

On $D^s(\mathcal{T}^n)$ one can introduce a strong Riemannian metric, say, as in [2, 4]. By $dist(\eta, \theta)$ we denote the Riemannian distance between η and θ (i.e., the infimum of curve lengths for curves joining η and θ). Introduce on $TD^s(\mathcal{T}^n)$ the distance d by the formula

$$d((X(\eta)), (Y(\theta))) = dist(\eta, \theta) + \|Q_e X(\eta) - Q_e Y(\theta)\|, \tag{1.1}$$

where the norm in $T_e D^s(\mathcal{T}^n)$ is generated by the strong Riemannian metric. Besides, we shall use the distance between the above-mentioned vectors in $\mathbb{R}^k$ after embedding. This distance is denoted by $\|iX(\eta) - iY(\theta)\|$.

Introduce another strong Riemannian metric on $TD^s_\mu(\mathcal{T}^n)$ as follows (see [3]). Represent the tangent space $T_{(m,X)}TD^s_\mu(\mathcal{T}^n)$ as the direct product of the vertical subspacde $\bar{V}_{(m,X)}$ and the space of Live-Civita connection $\bar{H}_{(m,X)}$ of the weak Riemannian metric (see [2]) on $D^s_\mu(\mathcal{T}^n)$. For every U and V

from $\bar{V}_{(m,X)}$ define the inner product as $(KU, KV)^s_\eta$ where K is the connector of the above-mentioned Levi-Civita connection and $(\cdot,\cdot)^s_\eta$ is the strong inner product in $T_\eta D^s(\mathcal{T}^n)$ generated by the strong Riemannian metric. For every X and Y from $\bar{H}_{(m,X)}$ define the inner product as $(T\pi X, T\pi Y)^s_\eta$. Set $\bar{H}_{(m,X)}$ and $\bar{V}_{(m,X)}$ to be orthogonal to each other. Thus, on $TD^s_\mu(\mathcal{T}^n)$ a certain strong Riemannian metric is well-defined. The Riemannian distance, i.e., the infimum of the length of curves, connecting the points in TD^s, with respect to the above Riemannian metric, is denoted by d_1.

Construct the distance $d_2(X,Y)$ on $TTD^s_\mu(\mathcal{T}^n)$, analogous to the distance d on $TD^s_\mu(\mathcal{T}^n)$, by the formula

$$d_2(X,Y) = d(\pi_1 X, \pi_1 Y) + \|Q_e K X_v - Q_e K Y_v\|$$
$$+\|Q_e T\pi X_h - Q_e T\pi Y_h\|, \tag{1.2}$$

where $\pi : TD^s_\mu(M) \to D^s_\mu(M)$ is the natural projection, X_v and Y_v are the vertical components of X and Y while X_h and Y_h are their horizontal components.

For the metrics $dist$, d, d_1 and d_2 and for the norm $\|\cdot\|$ in $T_e D^s(\mathcal{T}^n)$ we shall consider their Kuratowski measures of non-compactness which will be denoted by α_{dist}, α_d, α_{d_1}, α_{d_2} and $\alpha_{\|\cdot\|}$, respectively. We refer the reader, say, to [1], where the definitions of measures of non-compactness and of condensing operators are given and the corresponding theory is described in details.

We say that a force field $F(t,m,X)$ is given on a manifold M if in the tangent space $T_m M$ at every $m \in M$ a certain vector $F(t,m,X)$ depending on the time t and the vector $X \in T_m M$, is given. The force fields are right-hand sides of the second order differential equations on manifolds given in terms of covariant derivatives (see, e.g., [4]).

The main aim of the paper is investigation of set-valued force fields and the corresponding second order differential inclusions on the groups of diffeomorphisms of the flat n-dimensional torus with the use of the notion of parallelism. On this base some existence of solution theorems for second order differential inclusions on the above-mentioned groups are obtained.

The definitions and principal facts from the theory of set-valued maps and differential inclusions are contained in [6].

2 Second order differential inclusions

Lemma 1. *Let a set-valued force field $F : [0,l] \times TD^s(\mathcal{T}^n) \to TD^s(\mathcal{T}^n)$ with convex values satisfy the upper Caratheodory condition and be such that for almost all t for the mapping $A : [0,l] \times TD^s(\mathcal{T}^n) \to T_e D^s(\mathcal{T}^n)$ of the form $A(t,\eta,X) = Q_e F(t,\eta,X)$ and for every bounded set $\Omega \subset D^s(\mathcal{T}^n)$ the inequality $\alpha_{\|\cdot\|}(A(t,\Omega)) \le g(t)\,\alpha_d(\Omega)$ holds. Then for almost all t the vector* *field $F(t,\eta,X)$ is k-bounded with respect the measure of non-compactness α_d with the coefficient $1+g(t)$.*

Proof. By the hypothesis for every $\Omega \subset TD^s$, for which $\alpha_d(\Omega)$ is finite, the inequality $\alpha_{\|.\|}(A(t,\Omega)) \le g(t)\,\alpha_d(\Omega)$ holds. Specify $t \in [0,l]$. Suppose that $\alpha_d(\Omega)=\xi$, i.e., for every $\varepsilon>0$ there exists a finite cover of Ω by the sets Θ_i with diameters $\xi+\frac{\varepsilon}{2}$. Then from the hypothesis it follows that there exists a finite cover of $A(t,\Omega)\subset T_eD^s(\mathcal{T}^n)$ by the sets G_j with diameters $g(t)\,\xi+\frac{\varepsilon}{2}$. Consider the set $Q_\eta A(t,\Omega)\subset T_\eta D^s(\mathcal{T}^n)$. Then the set $= \bigcup\limits_{\eta\in\Omega} Q_\eta A(t,\Omega)$ has the natural structure of direct product $\Omega\times A(t,\Omega)$. Consider the set $G_{ij} = \bigcup\limits_{\eta\in\Theta_i} Q_\eta G_j$. The collection of sets G_{ij} forms a finite cover of Γ and the diameter of every such set with respect to the distance d is not greater then $\xi+g(t)\,\xi+\varepsilon$. Hence $\alpha_d() \le (1+g(t))\xi$. Since $F(t,\Omega)\subset\Gamma$, for almost all t the vector field $F(t,\eta,X)$ is k-bounded with respect to the measure of non-compactness α_d with the coefficient $1+g(t)$. □

Lemma 2. *Let the set-valued force field F on $TD^s(\mathcal{T}^n)$ is as in the previous Lemma. Then the vertical lift of this mapping*

$$F^l : [0,l]\times TD^s(\mathcal{T}^n) \to TTD^s(\mathcal{T}^n)$$

is k-bounded with respect to the measures of non-pcompactness α_d and α_{d_2} with the coefficient $2+g(t)$.

Proof. Specify $t\in[0,l]$. Consider the set $\Theta\subset TD^s(\mathcal{T}^n)$. Let its measure of non-compactness $\alpha_d(\Theta)=\xi$. This means that for every ε >0 it can be covered by a finite number of sets Θ_i with diameter $\xi+\varepsilon$. Then from the definition of distance d it follows that the set $\pi\Theta$ can be covered by a finite number of sets $\pi\Theta_i$ whose diameter is not greater than $\xi+\varepsilon$, i.e., $\alpha_d(\pi\Theta)\le\xi$. Then by the hypothesis $\alpha_{\|.\|}(A(t,\pi\Theta))\le g(t)\,\xi$, i.e., $(A(t,\pi\Theta))\subset T_eD^s(\mathcal{T}^n)$ can be covered by a finite number of sets G_j with the diameter nit greater than $g(t)\,\xi+\varepsilon$. By analogy with the proof of the previous Lemma consider the sets $G^l_{ij} = \bigcup\limits_{\theta\in\Theta_i}(Q_{\pi\theta}G_j)^l_\theta$. It is evident that the collection of all G^l_{ij} covers the image $F^l(t,\Theta)$. Since we have the finite number of those sets and the diameter of each one is not greater than $2\xi+g(t)\xi+\varepsilon$, the Lemma follows. □

Introduce the norm $\left|\left|\left|F^l\right|\right|\right| = \sup\limits_{y\in F^l}|||y|||$. Choose an arbitrary point $Z\in TD^s(\mathcal{T}^n)$. Since at every given t the set-valued map F^l is upper semicontinuous, there exists a neighborhood $V'(Z)\subset TD^s(\mathcal{T}^n)$ of the point Z such that for $Y\in V'(Z)$ the relation $\left|\left|\left|F^l(t,Y)\right|\right|\right| < \left|\left|\left|F^l(t,Z)\right|\right|\right| + C$ holds.

Determine the neighborhood $\tilde{V}(Z)\subset TD^s(\mathcal{T}^n)$ by the formula $\tilde{V} = V\bigcap V'$ where V is the neighborhood from [3, Theorem 1]. Specify a neighborhood

$D \subset U$ of Z as in [5, Theorem 1.4] such that $r(D) \subset \tilde{V}$. By [5, Theorem 1.4] the retraction r is Lipschitz continuous on D with the constant 2.

Theorem 3. *On the domain D for almost all t the vector field $\bar{F}^l$ is k-bounded with respect to the measure of non-compactness $\alpha_{\|\cdot\|}$ with the coefficient*

$$2\,(2+g(t))\left(1+a+k(C+\left|\left|\left|F^l\,(t,Z)\right|\right|\right|)\right).$$

Proof. Consider the set $\Omega \subset D$. Let $\alpha(\Omega) = \xi$. I.e., for every $\varepsilon > 0$ it can be covered by a finite number of sets Ω_i with diemeter non greater than $\xi + \varepsilon$. Consider the set $r(\Omega) \subset TD^s(\mathcal{T}^n)$, where r the retraction mentioned above. By [5, Theorem 1.4] the retraction r is Lipschitz continuous on D with the constant 2 with respect to the norm $\|\cdot\|$ on D and the distance d_1 on $TD^s(\mathcal{T}^n)$. Hence the set $r(\Omega)$ can be covered by a finite number of the sets with diameter not greater than $2\xi + \varepsilon$ wigth respect to d_1. Consider the set $F^l\,(t, r(\Omega)) \subset TTD^s\,(\mathcal{T}^n)$. This set can be covered by a finite number of sets with diameter not greater than $2\,(2+g(t))\,\xi + \varepsilon$ with respect to the distance d_2. Now from Lemma 2 and from the construction of the neighborhoods $\tilde{V}$ and of D it follows that the set $\bar{F}^l\,(t,\Omega)$ can be covered by a finite number of sets with diameter not greater tan $2\,(2+g(t))\left(1+a+k(C+\left|\left|\left|F^l\,(t,Z)\right|\right|\right|)\right)\xi + \varepsilon$. Hence $\bar{F}^l$ is condensing on D with respect to $\alpha_{\|cdot\|}$ with the coefficient

$$2\,(2+g(t))\left(1+a+k(C+\left|\left|\left|F^l\,(t,Z)\right|\right|\right|)\right). \qquad \square$$

Let F be a set-valued force field with convex images on $TD^s(\mathcal{T}^n)$ that satisfies the upper Caratheodory condition. Consider the differential inclusion

$$\frac{\tilde{D}}{dt}\dot{\eta}(t) \in F\,(t,\eta,\dot{\eta})\,. \tag{2.1}$$

This problem is reduced to the differential inclusion $\dot{\eta} \in \tilde{S} + F^l$ on $TD^s(\mathcal{T}^n)$ where F^l is the vertical lift of F TO $D^s(\mathcal{T}^n)$ and $\tilde{S}$ is the geodesic spray of the Levi-Civita connection of the weak metrics. It is known that $\tilde{S}$ is smooth and satisfies the condition $T_\pi\tilde{S}(X) = X$. Consider the extension $\bar{S}: U \to U$ of $\tilde{S}: TD^s(\mathcal{T}^n) \to TTD^s\,(\mathcal{T}^n)$ defined by the formula $\bar{S}(x) = TjS(r(x)),\ x \in U$.

Theorem 4. *Let the set-valued force field $F: [0,l] \times TD^s\,(\mathcal{T}^n) \to TD^s\,(\mathcal{T}^n)$ with convex images satisfy the upper Caratheodory condition and be such that for almost all t the map $A: [0,l] \times TD^s\,(\mathcal{T}^n) \to T_eD^s\,(\mathcal{T}^n)$ of the form $A\,(t,X) = Q_eF\,(t,X)$ is k-bounded with respect to the measures of non-compactness α_d and $\alpha_{\|\cdot\|}$ with the coefficient $g(t)$. Then for almost all t the vector field $\bar{S} + \bar{F}^l$ is locally k-bounded on a small enough neighborhood U in $TD^s(\mathcal{T}^n)$ with respect to the measures of non-compactness $\alpha_{|||\cdot|||}$.*

Proof. By Theorem 3 for almost all $t \in [0, l]$ for every $Z \in TD^s(\mathcal{T}^n)$ there exists its neighborhood D in U, on which the set-valued force field $\bar{F}^l$ is k-bounded with the coefficient $k = 2\,(2 + g(t))\left(1 + a + k(C + \left|\left|\left|F^l\,(t, Z)\right|\right|\right|)\right)$ relative to the measure of non-compactness $\alpha_{|||\cdot|||}$. The geodesic spray $\tilde{S}$ is a C^∞-smooth vector field. The embedding j and the retraction r are S^∞-smooth as well. Thus the vector field $\bar{S}$ on U is C^∞ smooth and so, in particular, locally Lipschitz continuous. Hence on a small enough neighborhood of the point Z the field $\bar{S}$ is Lipschitz continuous with a certain constant $g > 0$. Without loss of generality one can suppose that D is the above-mentioned neighborhood. By the properties of Kuratowski's measure of non-compactness the sum of locally k-bounded and a locally Lipschitz continuous field is locally k-bounded. Hence the set-valued vector field $\bar{S} + \bar{F}^l$ is locally k-bounded with respect to the measure of non-compacness $\alpha_{|||\cdot|||}$ with the coefficient $k = 2\,(2 + g(t))\left(1 + a + k(C + \left|\left|\left|F^l\,(t, Z)\right|\right|\right|)\right) + g$. □

Theorem 5. *Let the hypothesis of Theorem 4 are fulfilled and the function $g(t)$ be square integrable on the interval $[0, T]$. Specify a point $Z_0 \in TD^s(\mathcal{T}^n)$. Suppose that on the closure $\bar{D}$ of a certain neighborhood D of this point the estimate $\|F\,(t, X)\,\| < f(t)$, $X \in \bar{D}$ holds, where $f(t) > 0$ is a real function that is square integrable on $[0, l]$. Then the initial value problem (2.1) with initial condition $\eta(0) = \pi Z_o, \dot{\eta}(0) = Z_0$ has a local solution.*

Proof. Without loss of generality one can consider D as a neighborhood from the proof of Theorem 4. Consider the initial value problem $\gamma'(t) \in \bar{S} + \bar{F}^l$ on U with the initial condition $\gamma(0) \in Z_0 \in jT_eD^s(\mathcal{T}^n)$. Under the hypothesis, the function

$$k(t) = 2\,(2 + g(t))\left(1 + a + k(C + \left|\left|\left|F^l\,(t, Z)\right|\right|\right|)\right) + g$$

is integrable on $[0, l]$. Then from Theorem 4 it follows that the right-hand side of the latter differential inclusion satisfies the conditions of [6, Theorm 5.2.1] and so this initial value problem has a local solution. By analogy with [5, Theorm 2.4] one can easily prove that this solution belongs to $jTD^s(\mathcal{T}^n)$. Inclusion (2.1) is reduced to the inclusion with right-hand side $\tilde{S} + F^l$. This means that $\pi\gamma(t)$ satisfies inclusion (2.1). □

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