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# Examples of electromagnetic waves that admit subgroups of the Poincaré group

**M. A. Parinov**
Department of Mathematics, Ivanovo State University,
153025, Russia, Ivanovo, Ermaka str., 39
email: mihailparinov@mail.ru; map1951.ivgu@mail.ru

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**Abstract**

We present the result of classification of Maxwell spaces with zero current (electromagnetic waves) that admit subgroups of the Poincaré group. For $3 \leq k \leq 6$, we give examples of such spaces whose symmetry group is $k$-dimensional.

**keywords:** Minkowski space; Poincaré group; Maxwell equations; Maxwell space; electromagnetic wave; classification.

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## 1. Introduction

In classical theory, the electromagnetic field is described by a skew symmetric tensor $F_{ij}$ on a four-dimensional real manifold $M \subset \mathbf{R}_1^4$ (a domain in the Minkowski space) satisfying the Maxwell equations [1]

$$\partial_i F_{jk} + \partial_j F_{ki} + \partial_k F_{ij} = 0, \tag{1.1}$$
$$\nabla_k F^{ik} = -\tfrac{4\pi}{c} J^i \tag{1.2}$$

($i, j, k = 1, \ldots, 4$; the current $J^i$ must satisfy the equation $\nabla_i J^i = 0$)[1]

[1] $\nabla_k$ is the covariant derivative acting by the rule: $\nabla_k F^{ij} = \partial_k F^{ij} + \Gamma^i_{mk} F^{mj} + \Gamma^j_{mk} F^{im}$, $\nabla_k J^i = \partial_k J^i + \Gamma^i_{mk} J^m$, where $\Gamma^l_{ij} = \frac{1}{2} g^{kl}(\partial_i g_{kj} + \partial_j g_{ki} - \partial_k g_{ij})$.

We say that the *Maxwell space* is a triple $(M, g, F)$, where $M$ is a smooth, real, four-dimentional manifold, $F = \frac{1}{2}F_{ij}dx^i \wedge dx^j$ is a generalized symplectic structure[2] on $M$, and $g = g_{ij}dx^i dx^j$ is a pseudo-Euclidean metric on $M$ with the Lorentz signature $(- - -+)$.

The equation $dF = 0$ means that the form $F$ is closed and is equivalent to the first Maxwell equation. If the second Maxwell equation for the tensor $F_{ij}$ holds, then the Maxwell space corresponds to some electromagnetic field.

Let $G_g$ be the Poincaré group, i. e., the group of motions of the Minkowski space (or, equivalently, the group of diffeomorphisms of manifold $M$ that preserve the structure $(M, g)$). Further, let $G_F$ be the group of symplectomorphisms of the structure $(M, F)$. By $G_S$ we denote the group of diffeomorphisms of the manifold $M$ that preserves both $g$ and $F$, i. e., $G_S = G_g \cap G_F$. Note that $G_S$ is a subgroup of $G_g$. The Maxwell spaces with non-trivial groups $G_S$ are interesting, for example, in connection with the well-known method for obtaining the first integrals of the Lorentz equations [2].

The electromagnetic fields that admit the group $G_S$ were intensively studied in 1960–70s (see [3] – [8]). The relativistic symmetry groups (maximal subgroups of the Poincaré group that preserve tensor $F_{ij}$) were found in [3, 4, 5] for particular fields $F_{ij}$ (homogeneous fields, plane waves, etc.); the structure of these subgroups was studied. The connected subgroups of the Poincaré group that are invariant transformation groups of electromagnetic fields (relativistic symmetry groups) were studied in [6, 7, 8]. In particular, it was proved that the dimension of such a group is not greater than six [8]; the classification of these groups was obtained in [6, 7]. The problem of classification with respect to conjugation for connected subgroups of the Poincaré group was proposed in [9] without any reference to electrodynamics.

Solution for the classification problem of Maxwell spaces that admit subgroups of the Poincaré group was proposed by the author in [10, 11]; it was based on the classification obtained by I. V. Bel'ko [9]. The problem of group classification for Maxwell spaces without current (MSWC or electromagnetic waves) was considered by A. S. Ivanova and M. A. Parinov in [12] – [20]; there we described some classes $W_{k,l}$ of MSWC that admitted certain subgroups, $G_{k,l}$, of the Poincaré group, but the existence of representatives for every class was not proved.

Using the method proposed in [21], we construct representatives, if they exist, for every class $W_{k,l}$. Otherwise we prove that symmetry group $G_S$ for every Maxwell space of the class $W_{k,l}$ is wider, actually, than $G_{k,l}$. Note that, for 1- and 2-dimensional groups $G_{k,l}$, the latter is impossible, see [22, 23].

In this paper, we present examples of MSWC whose symmetry group $G_S$ is $k$-dimensional and coincides with $G_{k,l}$ $(3 \le k \le 6)$. We also announce the

[2] Symplectic structure must be closed ($dF = 0$) and non-degenerate ($\det F_{ij} \neq 0$); for generalized symplectic structure non-degeneracy is not required.

classification theorem for Maxwell spaces without current and give a sketch of its proof; its complete proof is extremely long and at the moment it exist only as a manuscript.

Suppose that $\mathcal{L}_g$ is the Lie algebra of vector fields that corresponds to the Poincaré group $G_g$, and $\{x^i\}$ are the Galilean coordinates. We choose the basis in $\mathcal{L}_g$ as follows:

$$\begin{aligned} &e_1 = (1,0,0,0),\ e_2 = (0,1,0,0),\ e_3 = (0,0,1,0),\ e_4 = (0,0,0,1),\\ &e_{12} = (-x^2,x^1,0,0),\ \ e_{13} = (x^3,0,-x^1,0),\ \ e_{23} = (0,-x^3,x^2,0),\\ &e_{14} = (x^4,0,0,x^1),\quad e_{24} = (0,x^4,0,x^2),\quad e_{34} = (0,0,x^4,x^3). \end{aligned}$$

Subgroup $G' \subset G_g$ is *conjugate* to subgroup $G$ if there exists an element $h \in G_g$ such that $G' = h^{-1}Gh$. We use the classification of connected subgroups of the Poincaré group up to conjugation obtained by I. V. Bel'ko [9]. His list of subalgebras $\mathcal{L}_{k,l} \subset \mathcal{L}_g$ for dimensions 1–6, where every $\mathcal{L}_{k,l}$ is a representative of the class of subalgebras conjugate to each other, is included in this paper as Appendix. By $L\{\xi_1, \ldots, \xi_k\}$ we denote the span of vectors $\xi_1, \ldots, \xi_k$. By $G_{k,l}$ we denote subgroup of the Poincaré group that corresponds to the Lie algebra $\mathcal{L}_{k,l}$.

For every group $G_{k,l}$ (algebra $\mathcal{L}_{k,l}$), we define the class $C_{k,l}$ of Maxwell spaces as follows. Tensor $F_{ij}$ defining this class is a solution of the first Maxwell equation (1.1) and is a solution of the invariance conditions for $F_{ij}$ with respect to $G_{k,l}$:

$$L_{\xi_\alpha} F_{ij} = 0 \quad (\alpha = 1, \ldots, k), \tag{1.3}$$

where the $\xi_\alpha$ are basis vectors in $\mathcal{L}_{k,l}$, and $L_{\xi_\alpha}$ is the Lie derivative. Classes $C_{k,l}$ were described in [10, 11].

The above-mentioned definition of classes $C_{k,l}$ yields the classification of Maxwell spaces by means of subgroups of the Poincaré group $G_g$. Indeed, if $G'$ is an arbitrary subgroup of the Poincaré group, then it is conjugate to a subgroup $G_{k,l}$. So, there is a coordinate transformation $x^i = A^i_{i'}x^{i'} + a^i$ such that $G' = A^{-1}G_{k,l}A$ for some $A \in G_g$. This means that the class $C_{k,l}$ of Maxwell spaces, defined by the tensor $F_{ij}$, transforms into the class $C'$, defined by the tensor

$$F_{i'j'} = F_{ij}A^i_{i'}A^j_{j'}; \tag{1.4}$$

the Maxwell spaces of class $C'$ admit subgroup $G'$.[3]

For every group $G_{k,l}$, we define the class $W_{k,l}$ of Maxwell spaces without current (MSWC) as follows: $W_{k,l}$ is a subclass of the class $C_{k,l}$ such that tensor $F_{ij}$ satisfies the second Maxwell equation with zero current

$$\nabla_k F^{ik} = 0. \tag{1.5}$$

[3] The latter means that $\tilde{x}^{i'} = B^{i'}_{k'}x^{k'} + b^{i'}$ for any $B \in G'$, and the matrix $\tilde{F}_{k'l'} = F_{i'j'}B^{i'}_{k'}B^{j'}_{l'}$ is equal to $F_{i'j'}$.

The classes $W_{k,l}$ are described in [12] – [23]. The description of classes $W_{k,l}$ yields the classification of Maxwell spaces without current (in particular, electromagnetic waves) in terms of subgroups of the Poincaré group as above.

For every Maxwell space $(M, g, F)$ of classes $C_{k,l}$ or $W_{k,l}$, the dimension of the symmetry group $G_S$ is $\geq k$. But, for some Maxwell spaces of these classes (sometimes, for all of them), their symmetry groups are broader than $G_{k,l}$. So, the following important question arises: does there exist in $C_{k,l}$ a Maxwell space with symmetry group $G_S = G_{k,l}$ (a representative of the class)? The same question is important for $W_{k,l}$ as well. We give the answer to this question for Maxwell spaces without current in Section .

# 2. Examples of Maxwell spaces without current

Here we give examples of Maxwell spaces without current admitting subgroups $G_{k,l}$ that may be symmetry groups for a given $F_{ij}$. We consider only the cases $3 \leq k \leq 6$. For examples of cases with $k = 1, 2$, we refer the reader to [22, 23].

## 2.1 Examples for 3-dimensional subgroups

**2.1.1.** *Class* $W_{3,1c}$. For $\mathcal{L}_{3,1c} = L\{e_1, e_3, e_2 + e_4\}$, we see that MSWC is defined by the tensor $F_{ij}$ of the form

$$F_{12} = \Phi, \;\; F_{13} = C_1, \;\; F_{14} = C_2 - \Phi, \;\; F_{23} = \Psi, \;\; F_{24} = C_3, \;\; F_{34} = C_4 + \Psi, \tag{2.1}$$

where $C_k = const$, while $\Phi = \Phi(x^2 - x^4)$ and $\Psi = \Psi(x^2 - x^4)$ are arbitrary functions.

**Example 2.1.** Let $\Phi = A \sin(x^2 - x^4)$, $\Psi = B \sin 2(x^2 - x^4)$, $C_1 = 0$, $C_2 = C$, $C_3 = 0$, $C_4 = D$; then

$$\begin{aligned} &F_{12} = A \sin(x^2 - x^4), \quad F_{13} = 0, \quad F_{14} = C - A \sin(x^2 - x^4), \\ &F_{23} = B \sin 2(x^2 - x^4), \;\; F_{24} = 0, \;\; F_{34} = D - B \sin 2(x^2 - x^4). \end{aligned} \tag{2.2}$$

*If $A \neq 0$, and $B \neq 0$, and $C^2 + D^2 \neq 0$, then MSWC, defined by the tensor (2.2), admits 3-dimensional group $G_S = G_{3,1c}$.*

**2.1.2.** *Class* $W_{3,2a}$. For $\mathcal{L}_{3,2a} = L\{e_{13} + \lambda e_2,\, e_1,\, e_3\}$ $(\lambda \neq 0)$, we have

$$\begin{aligned}
F_{12} &= A\cos\frac{x^2+x^4}{\lambda} + B\sin\frac{x^2+x^4}{\lambda} + C\cos\frac{x^2-x^4}{\lambda} + D\sin\frac{x^2-x^4}{\lambda},\\
F_{14} &= A\cos\frac{x^2+x^4}{\lambda} + B\sin\frac{x^2+x^4}{\lambda} - C\cos\frac{x^2-x^4}{\lambda} - D\sin\frac{x^2-x^4}{\lambda},\\
F_{23} &= -B\cos\frac{x^2+x^4}{\lambda} + A\sin\frac{x^2+x^4}{\lambda} - D\cos\frac{x^2-x^4}{\lambda} + C\sin\frac{x^2-x^4}{\lambda},\\
F_{34} &= B\cos\frac{x^2+x^4}{\lambda} - A\sin\frac{x^2+x^4}{\lambda} - D\cos\frac{x^2-x^4}{\lambda} + C\sin\frac{x^2-x^4}{\lambda},\\
F_{13} &= K, \quad F_{24} = L \quad (A, B, C, D, K, L = const).
\end{aligned} \tag{2.3}$$

*If 1)* $A \neq 0$ *and* $C \neq 0$ *or 2)* $B \neq 0$ *and* $D \neq 0$*, then* $G_S = G_{3,2a}$.

**2.1.3.** *Class* $W_{3,3}$. For $\mathcal{L}_{3,3} = L\{e_{13} + \mu e_4,\, e_1,\, e_3\}$ $(\mu \neq 0)$, we have

$$\begin{aligned}
F_{12} &= A\cos\frac{x^2+x^4}{\mu} + B\sin\frac{x^2+x^4}{\mu} + C\cos\frac{x^2-x^4}{\mu} + D\sin\frac{x^2-x^4}{\mu},\\
F_{14} &= A\cos\frac{x^2+x^4}{\mu} + B\sin\frac{x^2+x^4}{\mu} - C\cos\frac{x^2-x^4}{\mu} - D\sin\frac{x^2-x^4}{\mu},\\
F_{23} &= -B\cos\frac{x^2+x^4}{\mu} + A\sin\frac{x^2+x^4}{\mu} + D\cos\frac{x^2-x^4}{\mu} - C\sin\frac{x^2-x^4}{\mu},\\
F_{34} &= B\cos\frac{x^2+x^4}{\mu} - A\sin\frac{x^2+x^4}{\mu} + D\cos\frac{x^2-x^4}{\mu} - C\sin\frac{x^2-x^4}{\mu},\\
F_{13} &= K, \quad F_{24} = L \quad (A, B, C, D, K, L = const).
\end{aligned} \tag{2.4}$$

*If 1)* $A \neq 0$ *and* $C \neq 0$ *or 2)* $B \neq 0$ *and* $D \neq 0$*, then* $G_S = G_{3,3}$.

**2.1.4.** *Class* $W_{3,5}$. For $\mathcal{L}_{3,5} = L\{e_{24},\, e_1,\, e_3\}$, we have

$$\begin{aligned}
F_{12} &= \frac{Ax^2+Bx^4}{(x^2)^2-(x^4)^2}, \quad F_{13} = K, \quad F_{14} = -\frac{Bx^2+Ax^4}{(x^2)^2-(x^4)^2},\\
F_{23} &= \frac{Cx^2+Dx^4}{(x^2)^2-(x^4)^2}, \quad F_{24} = L, \quad F_{34} = \frac{Dx^2+Cx^4}{(x^2)^2-(x^4)^2},
\end{aligned} \tag{2.5}$$

where $A, B, C, D, K, L$ are constants.

**Example 2.2.** Let $B = C = 0$; then

$$\begin{aligned}
F_{12} &= \frac{Ax^2}{(x^2)^2-(x^4)^2}, \quad F_{13} = K, \quad F_{14} = -\frac{Ax^4}{(x^2)^2-(x^4)^2},\\
F_{23} &= \frac{Dx^4}{(x^2)^2-(x^4)^2}, \quad F_{24} = L, \quad F_{34} = \frac{Dx^2}{(x^2)^2-(x^4)^2}.
\end{aligned} \tag{2.6}$$

*If* $A^2 + D^2 \neq 0$*, then MSWC, defined by the tensor* (2.6)*, admits 3-dimensional group* $G_S = G_{3,5}$.

**2.1.5.** *Class* $W_{3,6a}$. For $\mathcal{L}_{3,6a} = L\{e_{24} + \lambda e_3,\ e_2,\ e_4\}$ $(\lambda \neq 0)$, we have

$$\begin{aligned} F_{12} &= \cosh\frac{x^3}{\lambda}\left(A\sin\frac{x^1}{\lambda} - B\cos\frac{x^1}{\lambda}\right) + \sinh\frac{x^3}{\lambda}\left(C\sin\frac{x^1}{\lambda} - D\cos\frac{x^1}{\lambda}\right), \\ F_{14} &= \sinh\frac{x^3}{\lambda}\left(B\cos\frac{x^1}{\lambda} - A\sin\frac{x^1}{\lambda}\right) + \cosh\frac{x^3}{\lambda}\left(D\cos\frac{x^1}{\lambda} - C\sin\frac{x^1}{\lambda}\right), \\ F_{23} &= \sinh\frac{x^3}{\lambda}\left(A\cos\frac{x^1}{\lambda} + B\sin\frac{x^1}{\lambda}\right) + \cosh\frac{x^3}{\lambda}\left(C\cos\frac{x^1}{\lambda} + D\sin\frac{x^1}{\lambda}\right), \\ F_{34} &= \cosh\frac{x^3}{\lambda}\left(A\cos\frac{x^1}{\lambda} + B\sin\frac{x^1}{\lambda}\right) + \sinh\frac{x^3}{\lambda}\left(C\cos\frac{x^1}{\lambda} + D\sin\frac{x^1}{\lambda}\right), \\ & F_{13} = K, \quad F_{24} = L \quad (A, B, C, D, K, L = const). \end{aligned} \tag{2.7}$$

**Example 2.3.** Let $B = C = D = 0$; then

$$\begin{aligned} F_{12} &= A\sin\frac{x^1}{\lambda}\cosh\frac{x^3}{\lambda}, \quad F_{13} = K, \quad F_{14} = -A\sin\frac{x^1}{\lambda}\sinh\frac{x^3}{\lambda}, \\ F_{23} &= A\cos\frac{x^1}{\lambda}\sinh\frac{x^3}{\lambda}, \quad F_{24} = L, \quad F_{34} = A\cos\frac{x^1}{\lambda}\cosh\frac{x^3}{\lambda}. \end{aligned} \tag{2.8}$$

*If* $A \neq 0$, *then* $G_S = G_{3,6a}$.

**2.1.6.** *Class* $W_{3,8}$. For $\mathcal{L}_{3,8} = L\{e_{12} - e_{14} + \lambda e_2,\ e_3,\ e_2 - e_4\}$, we have

$$\begin{aligned} F_{12} &= -\frac{a_4}{2}\left(\tilde{x}^2\right)^2 - a_3\tilde{x}^2 - \frac{a_4}{2\lambda^2}\tilde{x}^1 + a_6, \quad F_{13} = a_1\tilde{x}^2 + a_5, \quad F_{14} = F_{12} + a_4, \\ F_{23} &= \frac{a_1}{2}\left(\tilde{x}^2\right)^2 + a_5\tilde{x}^2 + \frac{a_1}{2\lambda^2}\tilde{x}^1 + a_2, \quad F_{24} = a_4\tilde{x}^2 + a_3, \quad F_{34} = -F_{23} - a_1, \end{aligned} \tag{2.9}$$

where $a_k = const$, and

$$\begin{aligned} \tilde{x}^1 &= 2\lambda x^1 + \left(x^2 + x^4\right)^2, \quad \tilde{x}^2 = \frac{x^2 + x^4}{\lambda}, \quad \tilde{x}^3 = x^3, \\ \tilde{x}^4 &= \lambda x^4 + x^1(x^2 + x^4) + \frac{1}{3\lambda}\left(x^2 + x^4\right)^3. \end{aligned} \tag{2.10}$$

**Example 2.4.** Let $a_4 = K$ in (2.9), and $a_k = 0$ for $k \neq 4$; substituting (2.10) for $\tilde{x}^i$ in (2.9), we obtain

$$\begin{aligned} F_{12} &= -\frac{K}{\lambda}x^1 - \frac{K}{\lambda^2}\left(x^2 + x^4\right)^2, \quad F_{13} = F_{23} = F_{34} = 0, \\ F_{14} &= F_{12} + K, \quad F_{24} = \frac{K}{\lambda}(x^2 + x^4). \end{aligned} \tag{2.11}$$

*If* $K \neq 0$, *then MSWC, defined by the tensor* (2.11), *admits 3-dimensional group* $G_S = G_{3,8}$.

**2.1.7.** *Example for the class $W_{3,9b}$ ($W_{3,9}$ for $\lambda = 0$ and $\mu \neq 0$).* The Lie algebra $\mathcal{L}_{3,9}$ takes the form $\mathcal{L}_{3,9b} = L\{e_{12} - e_{14} + \mu e_3,\ e_1,\ e_2 - e_4\}$.

*The Maxwell spaces without current, defined by the tensor*

$$\begin{aligned}
F_{12} &= F_{14} = -x^3 \left( b_1 \sin \frac{x^2 + x^4}{\mu} - b_2 \cos \frac{x^2 + x^4}{\mu} \right), \\
F_{13} &= \mu \left( b_1 \cos \frac{x^2 + x^4}{\mu} + b_2 \sin \frac{x^2 + x^4}{\mu} \right), \\
F_{23} &= -F_{34} = x^3 \left( b_1 \cos \frac{x^2 + x^4}{\mu} + b_2 \sin \frac{x^2 + x^4}{\mu} \right), \\
F_{24} &= \mu \left( b_1 \sin \frac{x^2 + x^4}{\mu} - b_2 \cos \frac{x^2 + x^4}{\mu} \right),
\end{aligned} \tag{2.12}$$

*admit the group $G_{3,9b}$; if $b_1 \neq 0$ or $b_2 \neq 0$, then $G_S = G_{3,9b}$.*

**2.1.8.** *Example for the class $W_{3,9c}$ ($W_{3,9}$ for $\lambda \neq 0$ and $\mu = 0$).* The Lie algebra $\mathcal{L}_{3,9}$ takes the form $\mathcal{L}_{3,9c} = L\{e_{12} - e_{14} + \lambda e_2,\ e_1,\ e_2 - e_4\}$.

*The Maxwell space without current, defined by the tensor*

$$\begin{aligned}
F_{12} &= \frac{K}{2\lambda^2}(x^2 + x^4)^2, \quad F_{13} = 0, \quad F_{14} = F_{12} + K, \\
F_{23} &= -F_{34} = \frac{K}{\lambda} x^3, \quad F_{24} = \frac{K}{\lambda}(x^2 + x^4),
\end{aligned} \tag{2.13}$$

*admits the group $G_{3,9c}$; if $K \neq 0$, then $G_S = G_{3,9c}$.*

**2.1.9.** *Example for the class $W_{3,10a}$ ($W_{3,10}$ for $\lambda \neq 0$ and $\mu \neq 0$).* We have

$$\mathcal{L}_{3,10a} = L\{e_{12} - e_{14} + \lambda e_2, e_1 + \mu e_3,\ e_2 - e_4\}.$$

*The Maxwell space without current, defined by the tensor*

$$\begin{aligned}
F_{12} &= -\frac{A(1 + 2\mu^2)}{2\lambda^2(1 + \mu^2)}(x^2 + x^4)^2 - \frac{\mu A}{\lambda(1 + \mu^2)}(\mu x^1 - x^3), \\
F_{13} &= 0, \quad F_{14} = F_{12} + A, \quad F_{24} = \frac{A}{\lambda}(x^2 + x^4), \\
F_{23} &= -F_{34} = -\frac{A}{2\lambda^2(1 + \mu^2)} \left[ 2\lambda\mu x^1 + \mu(x^2 + x^4)^2 - 2\lambda x^3 \right],
\end{aligned} \tag{2.14}$$

*admits the group $G_{3,10a}$; if $A \neq 0$, then $G_S = G_{3,10a}$.*

**2.1.10.** *Example for the class $W_{3,10b}$ ($W_{3,10}$ for $\lambda = 0$ and $\mu \neq 0$).* The Lie algebra $\mathcal{L}_{3,10}$ takes the form $\mathcal{L}_{3,10b} = L\{e_{12} - e_{14},\ e_1 + \mu e_3,\ e_2 - e_4\}$.

*The Maxwell space without current, defined by the tensor*

$$
\begin{aligned}
F_{12} = F_{14} &= \frac{K(\mu x^1 - x^3)}{\mu^2 (x^2 + x^4)^2} \sin \frac{\ln(x^2 + x^4)}{\mu}, \\
F_{13} &= \frac{K}{\mu (x^2 + x^4)} \cos \frac{\ln(x^2 + x^4)}{\mu}, \\
F_{23} = -F_{34} &= -\frac{K(\mu x^1 - x^3)}{\mu^2 (x^2 + x^4)^2} \cos \frac{\ln(x^2 + x^4)}{\mu}, \\
F_{24} &= \frac{K}{\mu (x^2 + x^4)} \sin \frac{\ln(x^2 + x^4)}{\mu},
\end{aligned}
\tag{2.15}
$$

*admits the group $G_{3,10b}$; if $K \neq 0$, then $G_S = G_{3,10b}$.*

**2.1.11.** *Example for the class $W_{3,11}$.* We have $\mathcal{L}_{3,11} = L\{e_{13} + \lambda e_{24},\ e_1,\ e_3\}$. *Tensor $F_{ij}$ of the form*

$$
\begin{aligned}
F_{12} &= \frac{A}{2\rho}\left(e^{\lambda\varphi}\cos\left(\varphi - \frac{\ln\rho}{\lambda}\right) - e^{-\lambda\varphi}\cos\left(\varphi + \frac{\ln\rho}{\lambda}\right)\right), \quad F_{13} = const, \\
F_{14} &= -\frac{A}{2\rho}\left(e^{\lambda\varphi}\cos\left(\varphi - \frac{\ln\rho}{\lambda}\right) + e^{-\lambda\varphi}\cos\left(\varphi + \frac{\ln\rho}{\lambda}\right)\right), \\
F_{23} &= \frac{A}{2\rho}\left(e^{\lambda\varphi}\sin\left(\varphi - \frac{\ln\rho}{\lambda}\right) - e^{-\lambda\varphi}\sin\left(\varphi + \frac{\ln\rho}{\lambda}\right)\right), \quad F_{24} = const, \\
F_{34} &= \frac{A}{2\rho}\left(e^{\lambda\varphi}\sin\left(\varphi - \frac{\ln\rho}{\lambda}\right) + e^{-\lambda\varphi}\sin\left(\varphi + \frac{\ln\rho}{\lambda}\right)\right),
\end{aligned}
\tag{2.16}
$$

*where*

$$
x^1 = r\cos(\theta - \varphi), \quad x^2 = \rho\cosh(\lambda\varphi), \quad x^3 = r\sin(\theta - \varphi), \quad x^4 = \rho\sinh(\lambda\varphi),
\tag{2.17}
$$

*defines MSWC that admits the group $G_{3,11}$; if $A \neq 0$, then $G_S = G_{3,11}$.*

**2.1.12.** *Class $W_{3,13}$.* For $\mathcal{L}_{3,13} = L\{e_{13},\ e_{24},\ e_2 - e_4\}$, we have

$$
\begin{aligned}
F_{12} = F_{14} &= -\frac{Lx^1 + Kx^3}{(x^2 + x^4)[(x^1)^2 + (x^3)^2]}, \quad F_{13} = M, \\
F_{23} = F_{34} &= \frac{Lx^3 - Kx^1}{(x^2 + x^4)[(x^1)^2 + (x^3)^2]}, \quad F_{24} = N,
\end{aligned}
\tag{2.18}
$$

where $K, L, M, N$ are constants.

*If $K \neq 0$ or $L \neq 0$, then MSWC, defined by the tensor* (2.18)*, admits 3-dimensional group $G_S = G_{3,13}$.*

**2.1.13.** *Class $W_{3,14}$.* For $\mathcal{L}_{3,14} = L\{e_{12} - e_{14} + \lambda e_1 + \mu e_3,\ e_{23} + e_{34} + \nu e_1 + \lambda e_3,\ e_2 - e_4\}$, we have

$$
\begin{aligned}
F_{12} = F_{14} &= -\varphi\,\Phi_1(u) + \psi\,\Phi_2(u) + \Phi_4(u), \quad F_{13} = \Phi_1(u), \\
F_{23} = -F_{34} &= \varphi\,\Phi_2(u) + \psi\,\Phi_1(u) + \Phi_3(u), \quad F_{24} = \Phi_2(u),
\end{aligned}
\tag{2.19}
$$

where

$$u = x^2 + x^4,\ \varphi = \frac{\mu x^1 + (u-\lambda)x^3}{u^2 - \lambda^2 + \mu\nu},\ \psi = \frac{\nu x^3 - (u+\lambda)x^1}{u^2 - \lambda^2 + \mu\nu}, \tag{2.20}$$

$\Phi_3(u)$ and $\Phi_4(u)$ are arbitrary functions, $\Phi_1(u)$ is a solution of the equation

$$\Phi_1'' + \frac{2u}{u^2 - \lambda^2 + \mu\nu}\Phi_1' - \frac{2u^2 + 2\lambda^2 + \mu^2 + \nu^2}{(u^2 - \lambda^2 + \mu\nu)^2}\Phi_1 = 0, \tag{2.21}$$

and

$$\Phi_2(u) = \frac{2u\Phi_1(u) + (u^2 - \lambda^2 + \mu\nu)\Phi_1'(u)}{\mu + \nu}. \tag{2.22}$$

**2.1.14.** *Class* $W_{3,16}$. For $\mathcal{L}_{3,16} = L\{e_{12} - e_{14},\ e_{24} + \lambda e_1 + \mu e_3,\ e_2 - e_4\}$, we have

$$\begin{aligned}
F_{12} = F_{14} &= \frac{A(x^3 - \mu\ln(x^2 + x^4)) + D(x^1 - \lambda\ln(x^2 + x^4)) + B}{x^2 + x^4},\\
F_{34} = -F_{23} &= \frac{A(x^1 - \lambda\ln(x^2 + x^4)) - D(x^3 - \mu\ln(x^2 + x^4)) + C}{x^2 + x^4},\\
F_{13} = A, \quad & F_{24} = D \quad (A,\ B,\ C,\ D = const).
\end{aligned} \tag{2.23}$$

*If* $A \neq 0$ *or* $D \neq 0$, *then* $G_S = G_{3,16}$.

**2.1.15.** *Class* $W_{3,17}$. For $\mathcal{L}_{3,17} = L\{e_{12} - e_{14},\ e_{23} + e_{34},\ e_{24}\}$, we have

$$\begin{aligned}
F_{12} &= \frac{2Ax^1x^3 + B\left[(x^1)^2 - (x^3)^2 + (x^2 + x^4)^2\right]}{2(x^2 + x^4)\left[(x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2\right]^2} + \frac{C}{x^2 + x^4},\\
F_{13} &= \frac{Bx^3 - Ax^1}{\left[(x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2\right]^2},\\
F_{14} &= \frac{2Ax^1x^3 + B\left[(x^1)^2 - (x^3)^2 - (x^2 + x^4)^2\right]}{2(x^2 + x^4)\left[(x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2\right]^2} + \frac{C}{x^2 + x^4},\\
F_{23} &= \frac{-2Bx^1x^3 + A\left[(x^1)^2 - (x^3)^2 - (x^2 + x^4)^2\right]}{2(x^2 + x^4)\left[(x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2\right]^2} + \frac{D}{x^2 + x^4},\\
F_{24} &= \frac{Bx^1 + Ax^3}{\left[(x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2\right]^2},\\
F_{34} &= \frac{2Bx^1x^3 - A\left[(x^1)^2 - (x^3)^2 + (x^2 + x^4)^2\right]}{2(x^2 + x^4)\left[(x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2\right]^2} - \frac{D}{x^2 + x^4},
\end{aligned} \tag{2.24}$$

where $A$, $B$, $C$, $D$ are constants.

*If* $A \neq 0$ *or* $B \neq 0$, *then* $G_S = G_{3,17}$.

**2.1.16.** *Class* $W_{3,18}$. For $\mathcal{L}_{3,18} = L\{e_{12} - e_{14},\, e_{23} + e_{34},\, e_{13} + \lambda(e_2 - e_4)\}$, we have

$$\begin{aligned} F_{12} &= -\frac{C_5}{2}\left(\left(\tilde{x}^2\right)^2 - \left(\tilde{x}^3\right)^2 + 1\right) - C_1\tilde{x}^2\tilde{x}^3 - C_4\tilde{x}^2 - C_8\tilde{x}^3, \\ F_{13} &= C_1\tilde{x}^2 - C_5\tilde{x}^3 + C_8, \quad F_{14} = F_{12} + C_5, \\ F_{23} &= \frac{C_1}{2}\left(\left(\tilde{x}^2\right)^2 - \left(\tilde{x}^3\right)^2 - 1\right) - C_5\tilde{x}^2\tilde{x}^3 + C_8\tilde{x}^2 - C_4\tilde{x}^3, \\ F_{24} &= C_5\tilde{x}^2 + C_1\tilde{x}^3 + C_4, \quad F_{34} = -F_{23} - C_1, \end{aligned} \tag{2.25}$$

where

$$\begin{aligned} C_1 &= 2C_{10} = \frac{1}{\tilde{x}^1}\left(A\cos\frac{\tilde{x}^4}{2\lambda\tilde{x}^1} + B\sin\frac{\tilde{x}^4}{2\lambda\tilde{x}^1}\right), \quad C_4 = \frac{L}{(\tilde{x}^1)^2}, \\ C_5 &= -2C_9 = \frac{1}{\tilde{x}^1}\left(-A\sin\frac{\tilde{x}^4}{2\lambda\tilde{x}^1} + B\cos\frac{\tilde{x}^4}{2\lambda\tilde{x}^1}\right), \quad C_8 = \frac{K}{(\tilde{x}^1)^2}, \end{aligned} \tag{2.26}$$

$A, B, K, L$ are constants, and

$$\tilde{x}^1 = x^2 + x^4, \;\; \tilde{x}^2 = -\frac{x^1}{x^2 + x^4}, \;\; \tilde{x}^3 = \frac{x^3}{x^2 + x^4}, \;\; \tilde{x}^4 = (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^4)^2. \tag{2.27}$$

*If* $A \neq 0$ *or* $B \neq 0$, *then* $G_S = G_{3,18}$.

**2.1.17.** *Class* $W_{3,19}$. For $\mathcal{L}_{3,19} = L\{e_{12} - e_{14},\, e_{23} + e_{34},\, e_{13} + \lambda e_{24}\}$, we have

$$\begin{aligned} F_{12} &= -\frac{C_5}{2}\left(\left(\tilde{x}^2\right)^2 - \left(\tilde{x}^3\right)^2 + 1\right) - C_1\tilde{x}^2\tilde{x}^3 + \varphi(\tilde{x}^1), \\ F_{13} &= C_1\tilde{x}^2 - C_5\tilde{x}^3, \quad F_{14} = F_{12} + C_5, \\ F_{23} &= \frac{C_1}{2}\left(\left(\tilde{x}^2\right)^2 - \left(\tilde{x}^3\right)^2 - 1\right) - C_5\tilde{x}^2\tilde{x}^3 + \psi(\tilde{x}^1), \\ F_{24} &= C_5\tilde{x}^2 + C_1\tilde{x}^3, \quad F_{34} = -F_{23} - C_1, \end{aligned} \tag{2.28}$$

where

$$C_1 = \frac{\tilde{x}^1}{(\tilde{x}^4)^2}\left[\left(A\cos\frac{\ln\tilde{x}^4}{\lambda} + B\sin\frac{\ln\tilde{x}^4}{\lambda}\right)\cos\frac{\ln\tilde{x}^1}{\lambda} + \left(A\sin\frac{\ln\tilde{x}^4}{\lambda} - B\cos\frac{\ln\tilde{x}^4}{\lambda}\right)\sin\frac{\ln\tilde{x}^1}{\lambda}\right],$$

$$C_5 = \frac{\tilde{x}^1}{(\tilde{x}^4)^2}\left[\left(A\sin\frac{\ln\tilde{x}^4}{\lambda} - B\cos\frac{\ln\tilde{x}^4}{\lambda}\right)\cos\frac{\ln\tilde{x}^1}{\lambda} - \left(A\cos\frac{\ln\tilde{x}^4}{\lambda} + B\sin\frac{\ln\tilde{x}^4}{\lambda}\right)\sin\frac{\ln\tilde{x}^1}{\lambda}\right],$$

$$\varphi(\tilde{x}^1) = \frac{1}{2\tilde{x}^1}\left(A\cos\frac{\ln\tilde{x}^1}{\lambda} + B\sin\frac{\ln\tilde{x}^1}{\lambda}\right),$$

$$\psi(\tilde{x}^1) = \frac{1}{2\tilde{x}^1}\left(A\sin\frac{\ln\tilde{x}^1}{\lambda} - B\cos\frac{\ln\tilde{x}^1}{\lambda}\right), \tag{2.29}$$

$A, B$ are constants, and the change of coordinates is defined by (2.27).

*If $A \neq 0$ or $B \neq 0$, then $G_S = G_{3,19}$.*

## 2.2 Examples for 4-dimensional subgroups

**2.2.1.** *Class $W_{4,4}$.* For $\mathcal{L}_{4,4} = L\{e_{13} + \lambda e_2,\ e_1,\ e_3,\ e_2 + e_4\}$ $(\lambda \neq 0)$, we have

$$F_{12} = -F_{14} = b_1\cos\frac{x^2 - x^4}{\lambda} - b_2\sin\frac{x^2 - x^4}{\lambda}, \quad F_{13} = b_3,$$
$$F_{23} = F_{34} = b_1\sin\frac{x^2 - x^4}{\lambda} + b_2\cos\frac{x^2 - x^4}{\lambda}, \quad F_{24} = b_4 \quad (b_i = const). \tag{2.30}$$

*Let $\lambda \neq 0$, $b_1^2 + b_2^2 \neq 0$, and $b_3^2 + b_4^2 \neq 0$; then MSWC, defined by the tensor (2.30), admits 4-dimensional group $G_S = G_{4,4}$.*

**2.2.2.** *Class $W_{4,5}$.* For $\mathcal{L}_{4,5} = L\{e_{24},\ e_1,\ e_3,\ e_2 + e_4\}$ we have

$$F_{12} = -F_{14} = \frac{b_1}{x^2 - x^4}, \quad F_{13} = b_3, \quad F_{23} = F_{34} = \frac{b_2}{x^2 - x^4}, \quad F_{24} = b_4 \quad . \tag{2.31}$$

$(b_k = const)$

*Let one of the following conditions be fulfilled: 1) $b_1 \neq 0$ and $b_3 \neq 0$, 2) $b_1 \neq 0$ and $b_4 \neq 0$, 3) $b_2 \neq 0$ and $b_3 \neq 0$, 4) $b_2 \neq 0$ and $b_4 \neq 0$; then $G_S = G_{4,5}$.*

**2.2.3.** *Class $W_{4,7}$.* For $\mathcal{L}_{4,7} = L\{e_{13} + \lambda e_{24},\ e_1,\ e_3,\ e_2 + e_4\}$ $(\lambda \neq 0)$, we have

$$-F_{12} = F_{14} = \frac{e^{\lambda\varphi}}{\rho}\left[K_1\sin\left(\varphi - \frac{\ln\rho}{\lambda}\right) + K_2\cos\left(\varphi - \frac{\ln\rho}{\lambda}\right)\right], \quad F_{13} = K_3,$$
$$F_{23} = F_{34} = \frac{e^{\lambda\varphi}}{\rho}\left[K_1\cos\left(\varphi - \frac{\ln\rho}{\lambda}\right) - K_2\sin\left(\varphi - \frac{\ln\rho}{\lambda}\right)\right], \quad F_{24} = K_4, \tag{2.32}$$

where $K_i = const$, and the change of coordinates is defined by (2.17).

*Let 1) $K_1^2 + K_2^2 \neq 0$ and 2) $K_3^2 + K_4^2 \neq 0$, then $G_S = G_{4,7}$.*

**2.2.4.** *Class $W_{4,9a}$.* For $\mathcal{L}_{4,9a} = L\{e_{12} - e_{14} + \lambda e_2,\ e_1,\ e_3,\ e_2 - e_4\}$ ($\lambda \neq 0$), we have

$$\begin{aligned} &F_{12} = F_{14} = -\frac{A}{\lambda}(x^2 + x^4) + D, \quad F_{13} = B, \quad F_{24} = A, \\ &F_{23} = -F_{34} = \frac{B}{\lambda}(x^2 + x^4) + C, \quad (A, B, C, D = const). \end{aligned} \tag{2.33}$$

*Let one of the following conditions be provided: 1) $A \neq 0$ and $C \neq 0$, 2) $D \neq 0$ and $B \neq 0$; then $G_S = G_{4,9a}$.*

**2.2.5.** *Class $W_{4,16}$.* For $\mathcal{L}_{4,16} = L\{e_{12} - e_{14} + \lambda e_3,\ e_{23} + e_{34} + \lambda e_1,\ e_{13},\ e_2 - e_4\}$, we have

$$\begin{aligned} &F_{12} = F_{14} = -\varphi\,\Phi_1(u) + \psi\,\Phi_2(u), \quad F_{13} = \Phi_1(u), \\ &F_{23} = -F_{34} = \varphi\,\Phi_2(u) + \psi\,\Phi_1(u), \quad F_{24} = \Phi_2(u), \end{aligned} \tag{2.34}$$

where

$$\begin{gathered} \Phi_1(u) = \frac{K}{u^2 + \lambda^2}, \\ \Phi_2(u) = \frac{Ku}{2\lambda(u^2 + \lambda^2)} + \frac{3Ku}{4\lambda^3} + \frac{3K(u^2 + \lambda^2)}{4\lambda^4} \arctan\frac{u}{\lambda}, \\ K = const, \end{gathered} \tag{2.35}$$

and

$$u = x^2 + x^4, \quad \varphi = \frac{\lambda x^1 + ux^3}{u^2 + \lambda^2}, \quad \psi = \frac{\lambda x^3 - x^1 u}{u^2 + \lambda^2}. \tag{2.36}$$

*If $K \neq 0$, then $G_S = G_{4,16}$.*

**2.2.6.** *Class $W_{4,18}$.* For $\mathcal{L}_{4,18} = L\{e_{12}, e_{13}, e_{23}, e_4\}$, we have

$$\begin{aligned} &F_{12} = F_{13} = F_{23} = 0, \quad F_{14} = \frac{Kx^1}{\rho^3}, \quad F_{24} = \frac{Kx^2}{\rho^3}, \quad F_{34} = \frac{Kx^3}{\rho^3} \\ &(\rho = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2},\ K = const). \end{aligned} \tag{2.37}$$

*If $K \neq 0$, then $G_S = G_{4,18}$.*

**2.2.7.** *Class $W_{4,19}$.* For $\mathcal{L}_{4,19} = L\{e_{12},\ e_{14},\ e_{24},\ e_3\}$, we have

$$\begin{aligned} &F_{12} = F_{14} = F_{24} = 0, \quad F_{13} = \frac{Kx^1}{u^3}, \quad F_{23} = \frac{Kx^2}{u^3}, \quad F_{34} = \frac{Kx^4}{u^3} \\ &(u = \sqrt{(x^1)^2 + (x^2)^2 - (x^4)^2},\ K = const). \end{aligned} \tag{2.38}$$

*If $K \neq 0$, then $G_S = G_{4,19}$.*

## 2.3 Examples for 5-dimensional and 6-dimensional subgroups

**2.3.1.** *Class* $W_{5,5}$. For $\mathcal{L}_{5,5} = L\{e_{12} - e_{14},\, e_{23} + e_{34} + \lambda e_2,\, e_1,\, e_3,\, e_2 - e_4\}$ $(\lambda = 0)$, we have

$$F_{12} = F_{14} = \Phi(x^2 + x^4), \quad F_{13} = F_{24} = 0, \quad F_{23} = -F_{34} = \Psi(x^2 + x^4), \quad (2.39)$$

where $\Phi(u)$ and $\Psi(u)$ are arbitrary functions.

**Example 2.5.** Suppose in (2.39) $\Phi = K\sin(x^2 + x^4)$ and $\Psi = L$ $(K, L = const)$. Then (2.39) takes the form

$$F_{12} = F_{14} = K\sin(x^2 + x^4),\ F_{13} = F_{24} = 0,\ F_{23} = -F_{34} = L. \quad (2.40)$$

*If $K \neq 0$ and $L \neq 0$, then $G_S = G_{5,5}$ $(\lambda = 0)$.*

**2.3.2.** *Class* $W_{6,2}$. For $\mathcal{L}_{6,2} = L\{e_{13},\, e_{24},\, e_1,\, e_2,\, e_3,\, e_4\}$, we have

$$F_{12} = F_{14} = F_{23} = F_{34} = 0,\ F_{13} = C_1,\ F_{24} = C_2\ (C_1, C_2 = const). \quad (2.41)$$

*If $C_1 \neq 0$ or $C_2 \neq 0$, then $G_S = G_{6,2}$.*

**2.3.3.** *Class* $W_{6,3}$. For $\mathcal{L}_{6,3} = L\{e_{12} - e_{14},\, e_{23} + e_{34},\, e_1,\, e_2,\, e_3,\, e_4\}$, we have

$$F_{12} = F_{14} = C_1,\ F_{23} = -F_{34} = C_2,\ F_{13} = F_{24} = 0\ (C_1, C_2 = const). \quad (2.42)$$

*If $C_1 \neq 0$ or $C_2 \neq 0$, then $G_S = G_{6,3}$.*

**2.3.4.** *Class* $W_{6,5}$. For $\mathcal{L}_{6,5} = L\{e_{12} - e_{14},\, e_{23} + e_{34},\, e_{13} + \lambda e_2,\, e_1,\, e_3,\, e_2 - e_4\}$, we have

$$\begin{aligned} &F_{12} = F_{14} = C_1 \sin\frac{x^2 + x^4}{\lambda} + C_2 \cos\frac{x^2 + x^4}{\lambda}, \quad F_{13} = F_{24} = 0, \\ &F_{23} = -F_{34} = C_1 \cos\frac{x^2 + x^4}{\lambda} - C_2 \sin\frac{x^2 + x^4}{\lambda} \quad (C_1,\, C_2 = const). \end{aligned} \quad (2.43)$$

*If $C_1 \neq 0$ or $C_2 \neq 0$, then $G_S = G_{6,5}$.*

**2.3.5.** *Class* $W_{6,6}$. For $\mathcal{L}_{6,6} = L\{e_{12} - e_{14},\, e_{23} + e_{34},\, e_{24},\, e_1,\, e_3,\, e_2 - e_4\}$, we have

$$F_{12} = F_{14} = \frac{K_1}{x^2 + x^4}, \quad F_{13} = F_{24} = 0, \quad F_{23} = -F_{34} = \frac{K_2}{x^2 + x^4}, \quad (2.44)$$

$(K_1,\, K_2 = const)$.

*If $K_1 \neq 0$ or $K_2 \neq 0$, then $G_S = G_{6,6}$.*

**2.3.6.** *Class* $W_{6,7}$. For $\mathcal{L}_{6,7} = L\{e_{12} - e_{14},\ e_{23} + e_{34},\ e_{13} + \lambda e_{24},\ e_1,\ e_3,\ e_2 - e_4\}$, we have

$$F_{12} = F_{14} = \frac{1}{x^2 + x^4}\left(a_1 \cos\frac{\ln(x^2 + x^4)}{\lambda} - a_2 \sin\frac{\ln(x^2 + x^4)}{\lambda}\right),$$
$$F_{13} = F_{24} = 0, \tag{2.45}$$
$$F_{23} = -F_{34} = \frac{1}{x^2 + x^4}\left(a_1 \sin\frac{\ln(x^2 + x^4)}{\lambda} + a_2 \cos\frac{\ln(x^2 + x^4)}{\lambda}\right),$$

$(a_1,\ a_2 = const)$.

*If* $a_1 \neq 0$ *or* $a_2 \neq 0$, *then* $G_S = G_{6,7}$.

# 3. The classification theorems of Maxwell spaces without current

**Theorem 3.1.** *(i) For every* 1*-dimensional or* 2*-dimensional subgroup of the Poincaré group* $G$*, there exist Maxwell spaces without current, such that* $G_S = G$.

*(ii) Classes of MSWC corresponding to subgroups* $G_{4,20}$, $G_{5,6}$, $G_{5,9}$, $G_{6,1}$, $G_{6,4}$, $G_{6,8}$, $G_{6,9}$ *are empty.*

*(iii) There do not exist MSWC with symmetry groups* $G_{3,1a}$, $G_{3,1b}$, $G_{3,2b}$ *(*$G_{3,2}$ *for* $\lambda = 0$*),* $G_{3,4}$, $G_{3,6b}$ *(*$G_{3,6}$ *for* $\lambda = 0$*),* $G_{3,7}$, $G_{3,9a}$ *(*$G_{3,9}$ *for* $\lambda = \mu = 0$*),* $G_{3,12}$, $G_{3,15}$, $G_{3,18b}$ *(*$G_{3,18}$ *for* $\lambda = 0$*),* $G_{3,20}$, $G_{3,21}$, $G_{4,1}$, $G_{4,2}$, $G_{4,3}$, $G_{4,4b}$ *(*$G_{4,4}$ *for* $\lambda = 0$*),* $G_{4,6}$, $G_{4,8}$, $G_{4,9b}$ *(*$G_{4,9}$ *for* $\lambda = 0$*),* $G_{4,10}$, $G_{4,11}$, $G_{4,12}$, $G_{4,13}$, $G_{4,14}$, $G_{4,15}$, $G_{4,17}$, $G_{4,20}$, $G_{5,1}$, $G_{5,2}$, $G_{5,3}$, $G_{5,4}$, $G_{5,5}$ *(for* $\lambda \neq 0$*),* $G_{5,6}$, $G_{5,7}$, $G_{5,8}$, $G_{5,9}$, $G_{6,1}$, $G_{6,4}$, $G_{6,8}$, *and* $G_{6,9}$.

*For the other subgroups* $G_{k,l}$*, there exist MSWC such that* $G_S = G_{k,l}$.

Sketch of the proof. Class $W_{k,l}$ is defined by the tensor $F_{ij}$ that is a solution of equations (1.1), (1.5) and (1.3) for basis vectors of algebra $\mathcal{L}_{k,l}$. Class $W_{k,l}$ is *empty*, if the system (1.1)–(1.5)–(1.3) has only trivial solution $F_{ij} = 0$; this is true for algebras listed in (ii).

To find the genuine symmetry group for Maxwell space of the class $W_{k,l}$ we must solve the equation

$$L_\xi F_{ij} = 0 \tag{3.1}$$

with respect to $\xi \in \mathcal{L}_g$[4]. We say that MSWC $(M, g, F)$ is a *representative* of the class $W_{k,l}$, if the space of solutions of equation $L_\xi F_{ij} = 0$ (Lie algebra $\mathcal{L}_S$) coincides with $\mathcal{L}_{k,l}$. For dimensions $k = 1$, 2, the representatives are found in

[4] Algebra $\mathcal{L}_g$ consists of the vectors $\xi^i = a^i_{.j} x^j + b^i$, where $a^i_{.j} = g^{ik} a_{kj}$ and $a_{kj} = -a_{jk}$, $b^i$ are real numbers.

[22, 23]. If the class $W_{k,l}$ corresponds to the group $G_{k,l}$ that belongs to the list in (iii), then it has no representatives ($\mathcal{L}_S \neq \mathcal{L}_{k,l}$ for every MSWC of $W_{k,l}$); for the other subgroups $G_{k,l}$, we described the representatives in Section 2.

**Theorem 3.2.** *(i) If subgroup $G \subset G_g$ conjugates to one of the following subgroups: $G_{4,20}$, $G_{5,6}$, $G_{5,9}$, $G_{6,1}$, $G_{6,4}$, $G_{6,8}$, $G_{6,9}$, then the class $W$ of MSWC, which admits subgroup $G$, is empty.*

*(ii) Subgroup $G \subset G_g$ cannot be a symmetry group for MSWC, if it is conjugate to one of the following subgroups: $G_{3,1a}$, $G_{3,1b}$, $G_{3,2b}$ ($G_{3,2}$ for $\lambda = 0$), $G_{3,4}$, $G_{3,6b}$ ($G_{3,6}$ for $\lambda = 0$), $G_{3,7}$, $G_{3,9a}$ ($G_{3,9}$ for $\lambda = \mu = 0$), $G_{3,12}$, $G_{3,15}$, $G_{3,18b}$ ($G_{3,18}$ for $\lambda = 0$), $G_{3,20}$, $G_{3,21}$, $G_{4,1}$, $G_{4,2}$, $G_{4,3}$, $G_{4,4b}$ ($G_{4,4}$ for $\lambda = 0$), $G_{4,6}$, $G_{4,8}$, $G_{4,9b}$ ($G_{4,9}$ for $\lambda = 0$), $G_{4,10}$, $G_{4,11}$, $G_{4,12}$, $G_{4,13}$, $G_{4,14}$, $G_{4,15}$, $G_{4,17}$, $G_{4,20}$, $G_{5,1}$, $G_{5,2}$, $G_{5,3}$, $G_{5,4}$, $G_{5,5}$ (for $\lambda \neq 0$), $G_{5,6}$, $G_{5,7}$, $G_{5,8}$, $G_{5,9}$, $G_{6,1}$, $G_{6,4}$, $G_{6,8}$, and $G_{6,9}$.*

Sketch of the proof. Let the class $W$ of MSWC, which admits subgroup $G$, be non-empty, and $G$ conjugate to one of subgroups $G_{k,l}$ listed in (i). Let the tensor $F_{ij} \neq 0$ generate MSWC of class $W$. Using (1.4) we get MSWC of class $W_{k,l}$, which corresponds to $G_{k,l}$ (see the paragraph above formula (1.4)). Therefore $W_{k,l}$ is not empty, in contradiction to the assumption.

Note that formula (1.4) sets non-generating reflection of classes MSWC. Otherwise this is a transformation of tensor $F_{ij}$ by some substitution of Galilean coordinates. Therefore symmetry groups have the same dimension as for $F_{ij}$ and for $F_{i'j'}$. As $G_S \neq G_{k,l}$ for classes $W_{k,l}$, which correspond to $G_{k,l}$ listed in (ii), then $\dim G'_S > k$ for conjugate subgroup $G'_S$.

## Conclusion

In this paper, we present the final results on the classification of Maxwell spaces without current by subgroups of Poincaré group. Using this classification we obtain many new wave solutions of Maxwell equations. Note that these solutions may describe electromagnetic waves but also may not: many of them are not electromagnetic waves as, for example, the static Maxwell spaces without current. We hope this work will be useful for applications.

## APPENDIX. Subgroups of the Poincaré group for dimensions 1–6 (Bel'ko's list)

### 1-dimensional subalgebras of $\mathcal{L}_g$

1) $\mathcal{L}_{1,1a} = L\{e_1\}$, $\mathcal{L}_{1,1b} = L\{e_4\}$, $\mathcal{L}_{1,1c} = L\{e_2 + e_4\}$;

2) $\mathcal{L}_{1,2} = L\{e_{13} + \lambda e_2 + \mu e_4\}$ ($\lambda, \mu = \text{const}$, $\lambda\mu(\lambda - \mu) = 0$);

3) $\mathcal{L}_{1,3} = L\{e_{24} + \lambda e_1\}$ ($\lambda = \text{const}$);

4) $\mathcal{L}_{1,4} = L\{e_{12} - e_{14} + \lambda e_2 + \mu e_3\}$ ($\lambda, \mu = \text{const}$, $\lambda\mu = 0$);

5) $\mathcal{L}_{1,5} = L\{e_{13} + \lambda e_{24}\}$ ($\lambda = \text{const} \neq 0$).

**2-dimensional subalgebras of $\mathcal{L}_g$**

1) $\mathcal{L}_{2,1a} = L\{e_1,\ e_2\}$, $\mathcal{L}_{2,1b} = L\{e_2,\ e_4\}$, $\mathcal{L}_{2,1c} = L\{e_1,\ e_2 + e_4\}$;

2) $\mathcal{L}_{2,2} = L\{e_{13} + \mu e_4,\ e_2\}$;

3) $\mathcal{L}_{2,3} = L\{e_{13} + \lambda e_2,\ e_4\}$;

4) $\mathcal{L}_{2,4} = L\{e_{13} + \lambda e_2,\ e_2 + e_4\}$;

5) $\mathcal{L}_{2,5} = L\{e_{24} + \lambda e_3,\ e_1\}$;

6) $\mathcal{L}_{2,6} = L\{e_{24} + \lambda e_3,\ e_2 - e_4\}$;

7) $\mathcal{L}_{2,7} = L\{e_{12} - e_{14} + \lambda e_2 + \mu e_3,\ e_2 - e_4\}$ ($\lambda\mu = 0$);

8) $\mathcal{L}_{2,8} = L\{e_{12} - e_{14} + \lambda e_2,\ e_3\}$;

9) $\mathcal{L}_{2,9} = L\{e_{13} + \lambda e_{24},\ e_2 - e_4\}$ ($\lambda \neq 0$);

10) $\mathcal{L}_{2,10} = L\{e_{13},\ e_{24}\}$;

11) $\mathcal{L}_{2,11} = L\{e_{12} - e_{14} + \lambda e_1 + \mu e_3,\ e_{23} + e_{34} - \mu e_1 + \lambda e_3\}$
($\lambda = 0,\ \mu \neq 0 \sim \lambda \neq 0, \mu = 0$);

12) $\mathcal{L}_{2,12} = L\{e_{12} - e_{14},\ e_{24} + \lambda e_3\}$.

**3-dimensional subalgebras of $\mathcal{L}_g$**

1) $\mathcal{L}_{3,1a} = L\{e_1, e_2, e_3\}$, $\mathcal{L}_{3,1b} = L\{e_1, e_2, e_4\}$,
$\mathcal{L}_{3,1c} = L\{e_1, e_3, e_2 + e_4\}$;

2) $\mathcal{L}_{3,2} = L\{e_{13} + \lambda e_2,\ e_1,\ e_3\}$;

3) $\mathcal{L}_{3,3} = L\{e_{13} + \mu e_4,\ e_1,\ e_3\}$ ($\mu \neq 0$);

4) $\mathcal{L}_{3,4} = L\{e_{13} + \lambda(e_2 + e_4),\ e_1,\ e_3\}$;

5) $\mathcal{L}_{3,5} = L\{e_{24},\ e_1,\ e_3\}$;

6) $\mathcal{L}_{3,6} = L\{e_{24} + \lambda e_3,\ e_2,\ e_4\}$;

7) $\mathcal{L}_{3,7} = L\{e_{24} + \lambda e_3,\ e_1,\ e_2 - e_4\}$;

8) $\mathcal{L}_{3,8} = L\{e_{12} - e_{14} + \lambda e_2,\ e_3,\ e_2 - e_4\}$;

9) $\mathcal{L}_{3,9} = L\{e_{12} - e_{14} + \lambda e_2 + \mu e_3,\ e_1,\ e_2 - e_4\}$ ($\lambda\mu = 0$);

10) $\mathcal{L}_{3,10} = L\{e_{12} - e_{14} + \lambda e_2,\ e_1 + \mu e_3,\ e_2 - e_4\}\ (\mu \neq 0)$;

11) $\mathcal{L}_{3,11} = L\{e_{13} + \lambda e_{24},\ e_1,\ e_3\}\ (\lambda \neq 0)$;

12) $\mathcal{L}_{3,12} = L\{e_{13} + \lambda e_{24},\ e_2,\ e_4\}$;

13) $\mathcal{L}_{3,13} = L\{e_{13},\ e_{24},\ e_2 - e_4\}$;

14) $\mathcal{L}_{3,14} = L\{e_{12} - e_{14} + \lambda e_1 + \mu e_3,\ e_{23} + e_{34} + \nu e_1 + \lambda e_3,\ e_2 - e_4\}$;

15) $\mathcal{L}_{3,15} = L\{e_{12} - e_{14},\ e_{24},\ e_3\}$;

16) $\mathcal{L}_{3,16} = L\{e_{12} - e_{14},\ e_{24} + \lambda e_1 + \mu e_3,\ e_2 - e_4\}$;

17) $\mathcal{L}_{3,17} = L\{e_{12} - e_{14},\ e_{23} + e_{34},\ e_{24}\}$;

18) $\mathcal{L}_{3,18} = L\{e_{12} - e_{14},\ e_{23} + e_{34},\ e_{13} + \lambda(e_2 - e_4)\}$;

19) $\mathcal{L}_{3,19} = L\{e_{12} - e_{14},\ e_{23} + e_{34},\ e_{13} + \lambda e_{24}\}\ (\lambda \neq 0)$;

20) $\mathcal{L}_{3,20} = L\{e_{12},\ e_{13},\ e_{23}\}$;

21) $\mathcal{L}_{3,21} = L\{e_{12},\ e_{14},\ e_{24}\}$.

**4-dimensional subalgebras of $\mathcal{L}_g$**

1) $\mathcal{L}_{4,1} = L\{e_1,\ e_2,\ e_3,\ e_4\}$;

2) $\mathcal{L}_{4,2} = L\{e_{13} + \lambda e_4,\ e_1,\ e_2,\ e_3\}$;

3) $\mathcal{L}_{4,3} = L\{e_{13} + \lambda e_2,\ e_1,\ e_3,\ e_4\}$;

4) $\mathcal{L}_{4,4} = L\{e_{13} + \lambda e_2,\ e_1,\ e_3,\ e_2 + e_4\}$;

5) $\mathcal{L}_{4,5} = L\{e_{24},\ e_1,\ e_3,\ e_2 + e_4\}$;

6) $\mathcal{L}_{4,6} = L\{e_{24} + \lambda e_3,\ e_1,\ e_2,\ e_4\}$;

7) $\mathcal{L}_{4,7} = L\{e_{13} + \lambda e_{24},\ e_1,\ e_3,\ e_2 + e_4\}\ (\lambda \neq 0)$;

8) $\mathcal{L}_{4,8} = L\{e_{12} - e_{14} + \lambda e_3,\ e_1,\ e_2,\ e_4\}$;

9) $\mathcal{L}_{4,9} = L\{e_{12} - e_{14} + \lambda e_2,\ e_1,\ e_3,\ e_2 - e_4\}$;

10) $\mathcal{L}_{4,10} = L\{e_{13},\ e_{24},\ e_1,\ e_3\}$;

11) $\mathcal{L}_{4,11} = L\{e_{13},\ e_{24},\ e_2,\ e_4\}$;

12) $\mathcal{L}_{4,12} = L\{e_{12} - e_{14} + \lambda e_3,\ e_{23} + e_{34} + \mu e_2,\ e_1,\ e_2 - e_4\}$;

13) $\mathcal{L}_{4,13} = L\{e_{12} - e_{14},\ e_{24} + \lambda e_1,\ e_3,\ e_2 - e_4\}$;

14) $\mathcal{L}_{4,14} = L\{e_{12} - e_{14},\ e_{24} + \lambda e_3,\ e_1 + \mu e_3,\ e_2 - e_4\}$;

15) $\mathcal{L}_{4,15} = L\{e_{12} - e_{14},\ e_{23} + e_{34},\ e_{24} + \lambda e_1,\ e_2 - e_4\}$;

16) $\mathcal{L}_{4,16} = L\{e_{12} - e_{14} + \lambda e_3,\ e_{23} + e_{34} + \lambda e_1,\ e_{13},\ e_2 - e_4\}$;

17) $\mathcal{L}_{4,17} = L\{e_{12} - e_{14},\ e_{23} + e_{34},\ e_{13} + \lambda e_{24},\ e_2 - e_4\}$ $(\lambda \neq 0)$;

18) $\mathcal{L}_{4,18} = L\{e_{12},\ e_{13},\ e_{23},\ e_4\}$;

19) $\mathcal{L}_{4,19} = L\{e_{12},\ e_{14},\ e_{24},\ e_3\}$;

20) $\mathcal{L}_{4,20} = L\{e_{12} - e_{14},\ e_{23} + e_{34},\ e_{13},\ e_{24}\}$.

**5-dimensional subalgebras of $\mathcal{L}_g$**

1) $\mathcal{L}_{5,1} = L\{e_{24},\ e_1,\ e_2,\ e_3,\ e_4\}$;

2) $\mathcal{L}_{5,2} = L\{e_{13} + \lambda e_{24},\ e_1,\ e_2,\ e_3,\ e_4\}$;

3) $\mathcal{L}_{5,3} = L\{e_{12} - e_{14},\ e_1,\ e_2,\ e_3,\ e_4\}$;

4) $\mathcal{L}_{5,4} = L\{e_{13},\ e_{24},\ e_1,\ e_3,\ e_2 + e_4\}$;

5) $\mathcal{L}_{5,5} = L\{e_{12} - e_{14},\ e_{23} + e_{34} + \lambda e_2,\ e_1,\ e_3,\ e_2 - e_4\}$;

6) $\mathcal{L}_{5,6} = L\{e_{12} - e_{14},\ e_{24} + \lambda e_3,\ e_1,\ e_2,\ e_4\}$;

7) $\mathcal{L}_{5,7} = L\{e_{12} - e_{14},\ e_{24},\ e_1,\ e_3,\ e_2 - e_4\}$;

8) $\mathcal{L}_{5,8} = L\{e_{12} - e_{14},\ e_{23} + e_{34},\ e_{24} + \lambda e_3,\ e_1,\ e_2 - e_4\}$;

9) $\mathcal{L}_{5,9} = L\{e_{12} - e_{14},\ e_{23} + e_{34},\ e_{13},\ e_{24},\ e_2 - e_4\}$.

**6-dimensional subalgebras of $\mathcal{L}_g$**

1) $\mathcal{L}_{6,1} = L\{e_{12},\ e_{13},\ e_{23},\ e_{14},\ e_{24},\ e_{34}\}$;

2) $\mathcal{L}_{6,2} = L\{e_{13},\ e_{24},\ e_1,\ e_2,\ e_3,\ e_4\}$;

3) $\mathcal{L}_{6,3} = L\{e_{12} - e_{14},\ e_{23} + e_{34},\ e_1,\ e_2,\ e_3,\ e_4\}$;

4) $\_6,4 = L\{e_{12} - e_{14},\ e_{24},\ e_1,\ e_2,\ e_3,\ e_4\}$;

5) $\mathcal{L}_{6,5} = L\{e_{12} - e_{14},\ e_{23} + e_{34},\ e_{13} + \lambda e_2,\ e_1,\ e_3,\ e_2 - e_4\}$;

6) $\mathcal{L}_{6,6} = L\{e_{12} - e_{14},\ e_{23} + e_{34},\ e_{24},\ e_1,\ e_3,\ e_2 - e_4\}$;

7) $\mathcal{L}_{6,7} = L\{e_{12} - e_{14},\ e_{23} + e_{34},\ e_{13} + \lambda e_{24},\ e_1,\ e_3,\ e_2 - e_4\}$;

8) $\mathcal{L}_{6,8} = L\{e_{12},\ e_{13},\ e_{23},\ e_1,\ e_2,\ e_3\}$;

9) $\mathcal{L}_{6,9} = L\{e_{12},\ e_{14},\ e_{24},\ e_1,\ e_2,\ e_4\}$.

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