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On projective classification of plane curves

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Abstract

Algebra of projective differential invariants and description of projective classes of regular smooth plane curves are found. A classification and projective normal forms for special classes of plane curves are given.

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1. Introduction

In this paper we investigate projective equivalence of smooth curves on the real projective plane. Some of these results are classical and known. Mainly they go back to Halphen's dissertation ([2]).

The considered curves are smooth but they have singularities in projective sense. For example, singular, from the projective point of view, are points on a curve, where tangent line has second order contact, i.e. inflection or *flex points*. Another example gives points on a curve, where osculating quadric has 5-th order contact, i.e. *Monge points*.

We describe $\mathbf{SL}_3(\mathbb{R})$-orbits of the projective action on jets of smooth plane curves up to 5-th order and classify all possible projective singularities. The level of 5-th jets taken for the only reason: starting from 6-jets regular orbits have trivial stabilizers, and from the level of 7-jets first differential invariants come up. This orbit classification gives projective classification, or projective normal forms for smooth curves, up to 6-th order jets.

To find the complete algebra of differential invariants we reproduce the Study derivation in suitable for us form. It allows us to prove that algebra of polynomial differential invariants which separates regular orbits can be obtained from the projective curvature by taking of higher Study derivatives (Theorem 4). As a by-product of this theorem we get two results: normal forms curves up to 10-th jets (Theorem 6) and projective classification of germs regular smooth curves (Theorem 5).

The rest of the paper devoted to special classes of plane curves. The popular and trivial classes plane curves such as straight lines and quadrics are singular from the projective point of view. The next class is the class of W-curves, introduced by Klein and Lie, we collected some known properties of these curves (see, [3] and [4] for more details). The more important property of these curves, in light of Theorem 5, is that they are not regular. They are curves of constant projective curvature.

The first regular curves can be found in cubics and theorem 16 repeats the known result that projective classes of regular cubics can be described by one parameter. We give explicit formula to find this parameter.

The second important class of regular curves delivers by extremals of Study functional (cf. [7]). The corresponding Euler equation has order 10 and therefore projective classes of such extremals can be described by two parameters. Theorem 16 gives a constructive way to compute them.

2. Jets of curves

Let $\mathbf{P}^2$ be the real projective plane and let $\mathbf{J}^k$ be the manifold of k - jets of non-parametrized curves on the plane $\mathbf{P}^2$. If $L \subset \mathbf{P}^2$ is a plane curve we denote by $[L]_a^k \in \mathbf{J}^k$ the k -jet of the curve at the point $a \in L$.

We denote by $\pi_{k,l} : \mathbf{J}^k \to \mathbf{J}^l,\ k > l$, the natural projections:

$$\pi_{k,l} : [L]_a^k \longmapsto [L]_a^l.$$

The structure of jet - manifolds can be described as follows: obviously $\mathbf{J}^0 = \mathbf{P}^2$, and the fibres $\pi_{1,0}^{-1}(a)\,,\ a \in \mathbf{P}^2$, of the projection $\pi_{1,0}$ are projectivizations of the tangent planes $\mathbb{P}(\mathbf{T}_a\mathbf{P}^2) = \mathbf{P}^1$, and fibres $\pi_{k,k-1}^{-1}\left([L]_a^{k-1}\right)$, when $k \geq 2$, are affine lines.

The vector spaces associated with them are

$$\mathbf{S}^k\tau_a^*\otimes\nu_a,$$

where $\tau_a^* = T_a^*L$ - cotangent space, and $\nu_a = T_a\mathbf{P}^2/T_aL$ - normal space to a curve.

Let (x,u) be an affine chart on the plane and let $(x,u,u_1,....,u_k)$ be the natural coordinates in the space of k -jets.

Here

$$u_i\left([L]_a^k\right) = \frac{\partial^i h}{\partial x^i}(b),$$

if $L = L_h \overset{\text{def}}{=} \{u = h(x)\}$ is a graph of function h in a neighborhood of point $a = (b, h(b))$.

In these coordinates the affine action is given by tensors

$$\theta = \frac{\lambda}{k!}dx^k\otimes\overline{\partial}_u\in\mathbf{S}^k\tau_a^*\otimes\nu_a,$$

and has the form

$$(x,u,u_1,...,u_{k-1},u_k)\longmapsto(x,u,u_1,...,u_{k-1},u_k+\lambda),$$

where $\overline{\partial}_u = \partial_u \mathrm{mod} T_aL$.

Any smooth curve $L\subset\mathbf{P}^2$ determines curves $L^{(k)}\subset\mathbf{J}^k$, k-th prolongations of L, formed by points $[L]_a^k$ where point a runs over curve L.

The action of projective group $\mathbf{SL}_3(\mathbb{R})$ can be prolonged in manifolds $\mathbf{J}^k$ in the natural way:

$$\phi^{(k)} : [L]_a^k \longmapsto [\phi(L)]_a^k$$

where ϕ is a projective transformation.

3. Model curves

The use of model curves is based on the following observation. Assume that we have a class $\mathfrak{M}$ of plane curves which is projectively invariant and such that for any point $x_k\in\mathbf{J}^k$ there is a unique curve $L = L(x_k)\in\mathfrak{M}$ such that $x_k = [L]_a^k$, for $a = \pi_k(x_k)$. Then points $x_{k+1} = [L(x_k)]_a^{k+1}$ can be taken as basic points in the affine line $\pi_{k+1,k}^{-1}(x_k)$ and the corresponding section $\mathfrak{m}:\mathbf{J}^k\to\mathbf{J}^{k+1}$ we consider as the zero section in the line bundle $\pi_{k+1,k}:\mathbf{J}^{k+1}\to\mathbf{J}^k$.

Let now $L\subset\mathbf{P}^2$ be an arbitrary curve. Then curves $L^{(k+1)}\subset\mathbf{J}^{k+1}$ and $\mathfrak{m}\left(L^{(k)}\right)\subset\mathbf{J}^{k+1}$ differs on element $\Theta_L\in\mathbf{S}^{k+1}T_L^*\otimes\nu_L$.

The last tensor is a projective differential invariant of order $(k+1)$ in the sense that

$$\phi^*\left(\Theta_{\phi(L)}\right) = \Theta_L,$$

for arbitrary projective transformation ϕ.

Below we'll realize this scheme for different classes of projective curves.

3.1 Straight Lines

Let $\mathfrak{M}$ be the class straight lines on the projective plane. Obviously, for any point $x_1 \in \mathbf{J}^1$ there is a unique line $L(x_1)$, such that $x_1 = [L(x_1)]_a^1$.

Therefore , the above construction leads us projective differential invariant of order 2

$$\Theta_{2L} \in \mathbf{S}^2 T_L^* \otimes \nu_L.$$

If $L = L_h$ is the graph of function $u = h(x)$ in the affine coordinates, then the restriction of tensor Θ_2 on this curve has the form:

$$\Theta_{2L} = h''(x)\ \frac{dx^2}{2!} \otimes \overline{\partial}_u.$$

We write

$$\Theta_2 = u_2\ \frac{dx^2}{2!} \otimes \overline{\partial}_u$$

in the jet coordinates and

$$\Theta_{2L} = \left.\Theta_2\right|_{L_h^{(2)}}.$$

Denote by $\Pi_2 \subset \mathbf{J}^2$ the submanifold, where $\Theta_2 = 0$.

Then the points

$$\Pi_2(L) = \Pi_2 \cap L^{(2)}$$

are precisely *inflection* or *flex points* on the curve, i.e. points where tangent lines have 2-rd order contact with the curve.

3.2 Quadrics

Let $\mathfrak{M}$ be now the class of quadrics on the projective plane.

In affine coordinates each such quadrics is defined by the equation:

$$Q_2 = a_{11}u^2 + 2a_{12}xu + a_{22}x^2 + 2a_1u + 2a_2x + a_3 = 0.$$

Taking derivatives of Q_2 up to order 5 and eliminating a's coefficients we arrive at equation

$$\left(9\,u_5{u_2}^2 + 40\,{u_3}^3 - 45\,u_2u_4u_3\right)u_2 = 0.$$

If we really will consider only quadrics ($u_2 \neq 0$), then we get the Monge equation:

$$9\,u_5{u_2}^2 + 40\,{u_3}^3 - 45\,u_2u_4u_3 = 0,$$

or

$$u_5 = \frac{5u_3u_4}{u_2} - \frac{40}{9}\frac{u_3^3}{u_2^2}.$$

In other words, for any point $x_4 \in \mathbf{J}^4 \setminus \pi_{4,1}^{-1}(\Pi_2)$ there is a unique quadric $Q(x_4)$ such that $[Q(x_4)]_a^4 = x_4$.

Therefore, as above, for any curve L we have projective differential invariant

$$\Theta_{5L} \in \mathbf{S}^5 T_L^* \otimes \nu_L,$$

where

$$\Theta_{5L} = \left(h^{(5)} - 5\frac{h^{(3)}h^{(4)}}{h^{(2)}} + \frac{40}{9}\frac{\left(h^{(3)}\right)^3}{\left(h^{(2)}\right)^2} \right) \frac{dx^5}{5!} \otimes \overline{\partial}_u,$$

or

$$\Theta_5 = \left(u_5 - \frac{5u_3u_4}{u_2} + \frac{40}{9}\frac{u_3^3}{u_2^2} \right) \frac{dx^5}{5!} \otimes \overline{\partial}_u$$

in jet coordinates in the domain $\mathbf{J}^5 \setminus \pi_{5,2}^{-1}(\Pi_2)$.

Denote by $\Pi_5 \subset \mathbf{J}^5 \setminus \pi_{5,2}^{-1}(\Pi_2)$ the submanifold, where $\Theta_5 = 0$.

Then the points

$$\Pi_5(L) = \Pi_5 \cap L^{(5)}$$

we call *Monge points.*

They are the points where osculating quadrics have 5-th order contact with the curve.

3.3 Cubics

Let $\mathfrak{M}$ be now the class of cubics on the projective plane.

In affine coordinates each such quadrics is defined by the equation:

$$Q_3 = a_{111}u^3 + 3a_{112}u^2x + 3a_{122}ux^2 + a_{222}x^3$$

$$+a_{11}u^2 + 2a_{12}xu + a_{22}x^2 + 2a_1u + 2a_2x + a_3 = 0.$$

Taking derivatives of Q_3 up to order 9 and eliminating a's coefficients we arrive at equation (see, for example, [6]):

$$u_2P_7u_9 + P_8 = 0,$$

where P_8 is a polynomial of degree 10 and order 8,

$$P_7 = 7\ (60)^{-3}\ \det(M_7)$$

is a a polynomial of degree 8 and order 7, and

$$M_7 = \left\| \begin{matrix} 120\,u_3 & 30\,u_4 & 6\,u_5 & u_6 & u_7/7 \\ 360\,u_2 & 120\,u_3 & 30\,u_4 & 6\,u_5 & u_6 \\ -180\,u_2{}^2 & 0 & 20\,u_3{}^2 & 10\,u_3u_4 & 2\,u_3u_5 + 5u_4{}^2/4 \\ 0 & 180\,u_2{}^2 & 120\,u_3u_2 & 30\,u_4u_2 + 20\,u_3{}^2 & 6\,u_5u_2 + 10\,u_3u_4 \\ 0 & 0 & 180\,u_2{}^2 & 180\,u_3u_2 & 60\,u_3{}^2 + 45\,u_4u_2 \end{matrix} \right\|$$

more explicitly

$$
\begin{aligned}
P_7 \;=\; & -33600\, u_2 {u_3}^6 u_4 - 810\, {u_2}^5 u_3 u_4 u_7 + 1134\, {u_2}^5 u_3 u_5 u_6 - 756\, {u_2}^4 {u_3}^2 {u_5}^2 \\
& +13230\, {u_2}^4 u_3 {u_4}^2 u_5 - 2835\, {u_2}^5 u_4 {u_5}^2 - 12600\, {u_2}^3 {u_3}^3 u_4 u_5 - 189\, {u_2}^6 {u_6}^2 \\
& -7875\, {u_2}^3 {u_3}^2 {u_4}^3 + 720\, {u_2}^4 {u_3}^3 u_7 - 4725\, {u_2}^4 {u_4}^4 + 11200\, {u_3}^8 \\
& +1890\, {u_2}^5 {u_4}^2 u_6 + 6720\, {u_2}^2 {u_3}^5 u_5 + 31500\, {u_2}^2 {u_3}^4 {u_4}^2 - 3150\, {u_2}^4 {u_3}^2 u_4 u_6 \\
& +162\, {u_2}^6 u_5 u_7.
\end{aligned}
$$

Therefore,

$$u_9 = -\frac{P_8}{u_2 P_7}$$

for cubics.

In other words, for any point $x_8 \in \mathbf{J}^8 \setminus \left(\pi_{8,2}^{-1}(\Pi_2) \cup \pi_{8,7}^{-1}(\Pi_7)\right)$, where $\Pi_7 = P_7^{-1}(0) \subset \mathbf{J}^7$, there is a unique cubic $Q(x_8)$ such that $[Q(x_x)]_a^8 = x_8$.

Therefore, as above, for any curve L we have projective differential invariant

$$\Theta_{9L} \in \mathbf{S}^9 T_L^* \otimes \nu_L,$$

where

$$\Theta_9 = \left(u_9 + \frac{P_8}{u_2 P_7}\right) \frac{dx^9}{9!} \otimes \overline{\partial}_u$$

in jet coordinates in the domain $\mathbf{J}^9 \setminus \left(\pi_{9,2}^{-1}(\Pi_2) \cup \pi_{9,7}^{-1}(\Pi_7)\right)$.

Denote by $\Pi_9 \subset \mathbf{J}^9 \setminus \left(\pi_{9,2}^{-1}(\Pi_2) \cup \pi_{9,7}^{-1}(\Pi_7)\right)$ the submanifold, where $\Theta_9 = 0$.

Then the points

$$\Pi_9(L) = \Pi_9 \cap L^{(9)}$$

we call *Monge cubic points.*

They are the points where osculating cubics have 9-th order contact with the curve.

3.4 General Polynomial forms

Let $\mathfrak{M}$ be now the class of curves on the projective plane, having degree n. Each such polynomial is defined by $k(k+3)/2$ coefficients. Hence, taking derivatives up to order $j = k(k+3)/2$ and eliminating these coefficients we arrive at equation of the form $P_j u_j + P_{j-1}$

4. $\mathbf{SL}_3(\mathbb{R})$-orbits in the jet spaces

4.1 $\mathbf{J}^2$-orbits

The action of the projective group on the manifold of 1-jets is obviously transitive.

The stabilizer of point $(0,0,0) \in \mathbf{J}^1$ is formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}.$$

Action of such matrices on the fibre of projection $\pi_{2,1} : \mathbf{J}^2 \to \mathbf{J}^1$ has the form:

$$A^{(2)} : (0,0,0,u_2) \longmapsto \left(0,0,0,a_{11}^{-3}\ u_2\right).$$

Therefore, there is the only one open regular orbit $\Pi_{20} = \mathbf{J}^2 \setminus \Pi_2$ and the singular orbit Π_2:

$$\mathbf{J}^2 = \Pi_{20} \cup \Pi_2,$$

and points

$$p_{20} = (0,0,0,1) \in \Pi_{20}, \quad p_2 = (0,0,0,0) \in \Pi_2$$

can be taken as representatives.

4.2 $\mathbf{J}^3$-orbits

Consider the action of the stabilizer of point $(0,0,0,1)$ from the regular orbit Π_{20} on the fibre of projection $\pi_{3,2} : \mathbf{J}^3 \to \mathbf{J}^2$.

This stabilizer is formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{11}^2 a_{33}^{-1} & 0 \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}$$

and their action is the following affine action:

$$A^{(3)} : (0,0,0,1,u_3) \longmapsto (0,0,0,1,\alpha_A\ u_3 + \beta_A),$$

where

$$\alpha_A = a_{33} a_{11}^{-1}, \quad \beta_A = 3\left(a_{11}a_{31} - a_{12}a_{33}\right) a_{11}^{-2}.$$

Therefore, $\Pi_{30} = \pi_{3,2}^{-1}(\Pi_{20})$ is the open regular orbit.

The stabilizer of the point $(0,0,0,0)$ from the singular orbit Π_2 formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}$$

which act in the following way

$$A^{(3)} : (0,0,0,0,u_3) \longmapsto \left(0,0,0,0,\ \frac{a_{33}}{a_{11}^4} u_3\right).$$

Therefore, the preimage $\pi_{3,2}^{-1}(\Pi_2)$ of the singular orbit is a union of two orbits $\Pi_{32} = \{(x, u, u_1, 0, 0)\}$ and $\Pi_{31} = \pi_{3,2}^{-1}(\Pi_2) \setminus \Pi_{31}$ and $\mathbf{J}^3$ has the following decomposition of $\mathbf{SL}_3(\mathbb{R})$-action:

$$\mathbf{J}^3 = \Pi_{30} \cup \Pi_{31} \cup \Pi_{32},$$

where Π_{30} is the regular open orbit.

The following points

$$p_{30} = (0,0,0,1,0) \in \Pi_{30}, \quad p_{31} = (0,0,0,0,1) \in \Pi_{31}, p_{32} = (0,0,0,0,0) \in \Pi_{32}$$

can be taken as representatives of the orbits.

4.3 $\mathbf{J}^4$-orbits

Consider the action of the stabilizer of point $(0,0,0,1,0)$ from the regular orbit Π_{30} on the fibre of projection $\pi_{4,3} : \mathbf{J}^4 \to \mathbf{J}^3$.

This stabilizer is formed by matrices

$$A = \left\| \begin{matrix} a_{11} & a_{11}a_{31}a_{33}^{-1} & 0 \\ 0 & a_{11}^2 a_{33}^{-1} & 0 \\ a_{31} & a_{32} & a_{33} \end{matrix} \right\|$$

with the following affine action:

$$A^{(4)} : (0,0,0,1,0,u_4) \longmapsto \left(0,0,0,1,0, a_{33}^2 a_{11}^{-2}\, u_4 + \left(6a_{33}a_{32} - 3a_{31}^2\right) a_{11}^{-2}\right).$$

Therefore, $\Pi_{40} = \pi_{4,3}^{-1}(\Pi_{30})$ is an open regular orbit.

The stabilizer of point $(0,0,0,0,1)$ from the singular orbit Π_{31} is formed by matrices

$$A = \left\| \begin{matrix} a_{11} & a_{12} & 0 \\ 0 & a_{11}^2 a_{33}^{-1} & 0 \\ a_{31} & a_{32} & a_{33} \end{matrix} \right\|$$

with the following affine action

$$A^{(4)} : (0,0,0,0,1,u_4) \longmapsto \left(0,0,0,0,1, a_{33}a_{11}^{-1}\, u_4 + 8a_{31}a_{11}^{-1}\right).$$

Therefore, $\Pi_{41} = \pi_{4,3}^{-1}(\Pi_{31})$ is an orbit.

Finally, the stabilizer of point $(0,0,0,0,0)$ from Π_{32} is formed by matrices

$$A = \left\| \begin{matrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{matrix} \right\|$$

with the following action

$$A^{(4)} : (0,0,0,0,0,u_4) \longmapsto \left(0,0,0,0,0, a_{33}^3 a_{22} a_{11}^{-4}\, u_4\right).$$

Therefore, the preimage $\pi_{4,3}^{-1}(\Pi_{32})$ of the singular orbit Π_{32} is a union of two orbits $\Pi_{43} = \{(x, u, u_1, 0, 0, 0)\}$ and $\Pi_{42} = \pi_{4,3}^{-1}(\Pi_{31}) \setminus \Pi_{43}$.

Summarizing, we see that there is the only one open and regular orbit Π_{40} and three singular orbits Π_{41}, Π_{42} and Π_{43} :

$$\mathbf{J}^4 = \Pi_{40} \cup \Pi_{41} \cup \Pi_{42} \cup \Pi_{43}.$$

The following points

$$\begin{aligned} p_{40} &= (0,0,0,1,0,0) \in \Pi_{40}, \quad p_{41} = (0,0,0,0,1,0) \in \Pi_{41}, \\ p_{42} &= (0,0,0,0,0,1) \in \Pi_{42}, p_{43} = (0,0,0,0,0,0) \in \Pi_{43} \end{aligned}$$

can be taken as representatives of the orbits.

4.4 $\mathbf{J}^5$-orbits

Let's begin with preimage of regular orbit Π_{40}.

The stabilizer of point $(0,0,0,1,0,0)$ is formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{11}a_{31}a_{33}^{-1} & 0 \\ 0 & a_{11}^2 a_{33}^{-1} & 0 \\ a_{31} & \frac{a_{31}^2 a_{33}}{2} & a_{33} \end{Vmatrix}$$

and has the following action on the fibre:

$$A^{(5)} : (0,0,0,1,0,0,u_5) \longmapsto \left(0,0,0,1,0,0, a_{33}^3 a_{11}^{-3}\ u_5\right).$$

Therefore, the preimage $\pi_{5,4}^{-1}(\Pi_{40})$ of the regular orbit is a union two orbits: the singular one Π_5 and the open regular orbit $\Pi_{50} = \pi_{5,2}^{-1}(\Pi_{20}) \setminus \Pi_5$.

The stabilizer of point $(0,0,0,0,1,0)$ from the singular orbit Π_{41} is formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{11}^3 a_{33}^{-2} & 0 \\ 0 & a_{32} & a_{33} \end{Vmatrix}$$

and acts in the following way

$$A^{(5)} : (0,0,0,0,1,0,u_5) \longmapsto \left(0,0,0,0,1,0, a_{33}^2 a_{11}^{-2}\ u_5 - 10 a_{12} a_{33}^2 a_{11}^{-3}\right).$$

Therefore, $\Pi_{51} = \pi_{5,4}^{-1}(\Pi_{41})$ is an orbit.

The stabilizer of point $(0,0,0,0,0,1)$ from the singular orbit Π_{42} is formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{11}^4 a_{33}^{-3} & 0 \\ a_{31} & a_{32} & a_{33} \end{Vmatrix}$$

and acts in the following way

$$A^{(5)}:(0,0,0,0,0,1,u_5)\longmapsto\left(0,0,0,0,0,1,a_{33}a_{11}^{-1}\ u_5+15a_{31}a_{11}^{-1}\right).$$

Therefore,$\Pi_{52}=\pi_{5,4}^{-1}\left(\Pi_{42}\right)$ is an orbit too.

Finally, the stabilizer of point $(0,0,0,0,0,0)$ from the singular orbit Π_{43} is formed by matrices

$$A=\begin{Vmatrix}a_{11} & a_{12} & 0\\ 0 & a_{22} & 0\\ a_{31} & a_{32} & a_{33}\end{Vmatrix}$$

and acts as following

$$A^{(5)}:(0,0,0,0,0,0,u_5)\longmapsto\left(0,0,0,0,0,1,a_{33}^3a_{11}^{-6}\ u_5\right).$$

Therefore, the preimage $\pi_{5,4}^{-1}\left(\Pi_{43}\right)$ is a union of two orbits:

$$\Pi_{54}=\{(x,u,u_1,0,0,0,0)\}$$

and

$$\Pi_{53}=\pi_{5,4}^{-1}\left(\Pi_{43}\right)\setminus\Pi_{54}.$$

Summarizing, we conclude that $\mathbf{SL}_3\left(\mathbb{R}\right)$-action in $\mathbf{J}^5$ has the following orbit decomposition:

$$\mathbf{J}^5=\Pi_{50}\cup\Pi_5\cup\Pi_{51}\cup\Pi_{52}\cup\Pi_{53}\cup\Pi_{54},$$

where Π_{50} is the only regular open orbit.

The following points

$$\begin{aligned}p_{50}&=(0,0,0,1,0,0,1)\in\Pi_{50},\quad p_5=(0,0,0,1,0,0,0)\in\Pi_5,\\ p_{51}&=(0,0,0,0,1,0,0)\in\Pi_{51},\quad p_{52}=(0,0,0,0,0,1,0)\in\Pi_{52},\\ p_{53}&=(0,0,0,0,0,0,1)\in\Pi_{53},\ p_{54}=(0,0,0,0,0,0,0)\in\Pi_{54}\end{aligned}$$

can be taken as representatives of the orbits.

4.5 $\mathbf{J}^6$-orbits

Let's begin with preimage of regular orbit Π_{50}. The stabilizer of point p_{50} is formed by matrices

$$A=\begin{Vmatrix}a_{33} & a_{31} & 0\\ 0 & a_{33} & 0\\ a_{31} & \frac{1}{2}a_{31}a_{33}^{-1} & a_{33}\end{Vmatrix}$$

which act in the following way

$$A^{(6)}:(0,0,0,1,0,0,1,u_6)\longmapsto\left(0,0,0,1,0,0,1,u_6+\frac{3a_{31}}{a_{33}}\right),$$

and therefore $\Pi_{60} = \pi_{6,5}\left(\Pi_{50}\right)$ is an open regular orbit.

The stabilizer of the point p_5 is formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{11}a_{31}a_{33}^{-1} & 0 \\ 0 & a_{11}^2 a_{33}^{-1} & 0 \\ a_{31} & \frac{1}{2}a_{31}^2 a_{33}^{-1} & a_{33} \end{Vmatrix}$$

with the following action:

$$A^{(6)} : (0,0,0,1,0,0,0,u_6) \longmapsto \left(0,0,0,1,0,0,0,a_{33}^4 a_{11}^{-4} u_6\right).$$

Therefore, preimage $\pi_{6,5}\left(\Pi_5\right)$ is a union of three orbits

$$\pi_{6,5}\left(\Pi_5\right) = \Pi_{61}^{+} \cup \Pi_{61}^{-} \cup \Pi_{62}$$

with the following representatives

$$p_{61}^{+} = (0,0,0,1,0,0,0,1)\,,\ p_{61}^{-} = (0,0,0,1,0,0,0,-1)\,,$$

$$p_{62} = (0,0,0,1,0,0,0,0)\,.$$

The stabilizer of the point $p_{51} = (0,0,0,0,1,0,0) \in \Pi_{51}$ contains matrices

$$A = \begin{Vmatrix} a_{11} & 0 & 0 \\ 0 & a_{11}^3 a_{33}^{-2} & 0 \\ 0 & a_{32} & a_{33} \end{Vmatrix}$$

and acts in the following way

$$A^{(6)} : (0,0,0,0,1,0,0,u_6) \longmapsto \left(0,0,0,1,0,0,0,a_{33}^3 a_{11}^{-3} u_6 + 40 a_{11}^{-3} a_{32} a_{33}^2\right).$$

Therefore, $\Pi_{63} = \pi_{6,5}^{-1}\left(\Pi_{51}\right)$ is an orbit.

The stabilizer of the point $p_{52} = (0,0,0,0,0,1,0) \in \Pi_{52}$ formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{11}^4 a_{33}^{-3} & 0 \\ 0 & a_{32} & a_{33} \end{Vmatrix}$$

and acts in the following way

$$A^{(6)} : (0,0,0,0,0,1,0,u_6) \longmapsto \left(0,0,0,1,0,0,0,a_{33}^2 a_{11}^{-2} u_6\right).$$

Therefore, $\pi_{6,5}^{-1}\left(\Pi_{52}\right)$ is a union of three orbits

$$\pi_{6,5}^{-1}\left(\Pi_{52}\right) = \Pi_{64}^{+} \cup \Pi_{64}^{-} \cup \Pi_{65}$$

with representatives

$$p_{64}^{+} = (0,0,0,0,0,1,0,1)\,,\ p_{64}^{-} = (0,0,0,0,0,1,0,-1)\,,$$

$$p_{65} = (0,0,0,0,0,1,0,0)\,.$$

The stabilizer of the point $p_{53} = (0,0,0,0,0,0,1) \in \Pi_{53}$ formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{11}^{5}a_{33}^{-4} & 0 \\ a_{13} & a_{32} & a_{33} \end{Vmatrix}$$

with the following action

$$A^{(6)} : (0,0,0,0,0,0,1,u_6) \longmapsto \left(0,0,0,0,0,0,1,a_{33}a_{11}^{-1}u_6 + 24\ a_{31}a_{11}^{-1}\right).$$

Therefore, $\Pi_{66} = \pi_{6,5}^{-1}(\Pi_{53})$ is an orbit with representative

$$p_{66} = (0,0,0,0,0,0,1,0)\,.$$

Finally, the stabilizer of the point $p_{54} = (0,0,0,0,0,0,1) \in \Pi_{54}$ is formed by matrices

$$A = \begin{Vmatrix} a_{11} & a_{12} & 0 \\ 0 & a_{22} & 0 \\ a_{13} & a_{32} & a_{33} \end{Vmatrix},$$

which act in the following way:

$$A^{(6)} : (0,0,0,0,0,0,0,u_6) \longmapsto \left(0,0,0,1,0,0,0,a_{33}^{4}a_{11}^{7}u_6\right).$$

Therefore, the preimage $\pi_{6,5}^{-1}(\Pi_{54})$ is a union of two orbits Π_{67} and Π_{68} with representatives

$$p_{67} = (0,0,0,0,0,0,0,1) \text{ and } p_{68} = (0,0,0,0,0,0,0,0)$$

respectively.

Summarizing, we get the following result.

Theorem 4.1. $\mathbf{SL}_3(\mathbb{R})$*-action in* $\mathbf{J}^6$ *splits into the following orbit decomposition:*

$$\mathbf{J}^6 = \Pi_{60} \cup \Pi_{61}^{+} \cup \Pi_{61}^{-} \cup \Pi_{62} \cup \Pi_{63} \cup \Pi_{64}^{+} \cup \Pi_{64}^{-} \cup \Pi_{65} \cup \Pi_{66} \cup \Pi_{67} \cup \Pi_{68},$$

with the following representatives

$$\begin{aligned} p_{60} &= (0,0,0,1,0,0,1,0), \quad p_{61}^{+} = (0,0,0,1,0,0,0,1)\,, \\ p_{61}^{-} &= (0,0,0,1,0,0,0,-1)\,, \ p_{62} = (0,0,0,1,0,0,0,0)\,, \\ p_{63} &= (0,0,0,0,1,0,0,0)\,, \quad p_{64}^{+} = (0,0,0,0,0,1,0,1)\,, \\ p_{64}^{-} &= (0,0,0,0,0,1,0,-1)\,, \ p_{65} = (0,0,0,0,0,1,0,0)\,, \\ p_{66} &= (0,0,0,0,0,0,1,0)\,, \quad p_{67} = (0,0,0,0,0,0,0,1)\,, \\ p_{68} &= (0,0,0,0,0,0,0,0)\,. \end{aligned}$$

As a corollary of this theorem we get the following $\mathbf{SL}_3(\mathbb{R})$ −classification of 6-jets of projective curves.

Theorem 4.2. *Let $L \subset \mathbf{P}^2$ be a smooth projective curve. Then for any point $a \in L$ there are projective coordinates (x, y) such that $x(a) = y(a) = 0$ and the curve can be written in the form $y = p(x) + \varepsilon(x)$, where function $\varepsilon(x)$ has seventh order of smallness and polynomial $p(x)$ has one of the following form:*

$$\begin{aligned} p_{60}(x) &= x^2 + x^5, \quad p_{61}^{\pm}(x) = x^2 \pm x^6, \quad p_{64}^{\pm} = x^4 \pm x^6, \\ p_{62}(x) &= x^2, \quad p_{63}(x) = x^3, \quad p_{65} = x^4, \quad p_{66} = x^5, \quad p_{67} = x^6, \quad p_{68} = 0, \end{aligned}$$

where polynomials p_{ij} correspond to orbits Π_{ij}.

4.6 Stabilizers of regular orbit

The open orbit $\Pi_{60} = \mathbf{J}^6 \setminus \pi_{6,2}(\Pi_2) \setminus \pi_{6,5}(\Pi_5)$ we call *regular,* and elements of this orbit we also call *regular.* A point $a \in L$ on a smooth projective curve we call *regular* if $[L]_a^6 \in \Pi_{60}$, if not the point is calling *singular.* It was to note that our definitions differ from the standard ones: both regular and singular points belong to smooth curve, and their singularity has projective nature .Remark also that the previous theorem states that the regular orbit is connected even though singular orbits Π_2 and Π_5 have codimension 1.

Before to consider differential invariants of projective curves we'll finish this section by description of stabilizers of regular point in $\mathbf{J}^k$, when $k = 2, 3, 4, 5, 6$.

Take 2-jet $p_{20} = (0, 0, 0, 1)$. Then the stabilizer is a 4-dimensional group and consist of matrices

$$\mathbf{St}_2 = \left\{ \begin{Vmatrix} 1 & \alpha & 0 \\ 0 & \beta^{-1} & 0 \\ \gamma & \delta & \beta \end{Vmatrix} \right\},$$

where $(\alpha, \gamma, \delta) \in \mathbb{R}^3, \beta \in \mathbb{R} \backslash 0$.

For 3-jet $p_{30} = (0, 0, 0, 1, 0)$ the stabilizer is a 3-dimensional group and consist of matrices

$$\mathbf{St}_3 = \left\{ \begin{Vmatrix} 1 & \alpha & 0 \\ 0 & \beta^{-1} & 0 \\ \alpha\beta & \gamma & \beta \end{Vmatrix} \right\},$$

where $(\alpha, \gamma) \in \mathbb{R}^2, \beta \in \mathbb{R} \backslash 0$.

For 4-jet $p_{40} = (0, 0, 0, 1, 0, 0)$ the stabilizer is a 2-dimensional group and consist of matrices

$$\mathbf{St}_4 = \left\{ \begin{Vmatrix} 1 & \alpha & 0 \\ 0 & \beta^{-1} & 0 \\ \alpha\beta & \frac{1}{2}\alpha^2\beta & \beta \end{Vmatrix} \right\},$$

where $\alpha \in \mathbb{R}, \beta \in \mathbb{R}\backslash 0$.

For 5-jet $p_{50} = (0,0,0,1,0,0,1)$ the stabilizer is a 1-dimensional group and consist of matrices

$$\mathbf{St}_5 = \left\{ \left\| \begin{matrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ \alpha & \frac{1}{2}\alpha^2 & 1 \end{matrix} \right\| \right\},$$

and for 6-jet $p_{60} = (0,0,0,1,0,0,1,0)$ the stabilizer is trivial.

5. Projective Invariants

5.1 Relative Invariants

As we have seen, functions

$$\begin{aligned} P_2 &= u_2, \\ P_5 &= u_5 - \frac{5u_3u_4}{u_2} + \frac{40}{9}\frac{u_3^3}{u_2^2} \end{aligned}$$

determine singular orbits Π_2 and Π_5.

Therefore, they are relative invariants of the $\mathbf{SL}_3(\mathbb{R})$- action.

Indeed, it is easy to check that

$$X^{(2)}(P_2) = \alpha_2(X) \cdot P_2,$$

where

$$\begin{aligned} X = & \left(2\,a_{1,1}x + a_{2,2}x + a_{1,2}u + a_{1,3} - a_{3,1}x^2 - a_{3,2}xu\right)\partial_x \\ & + \left(a_{1,1}u + 2\,a_{2,2}u + a_{2,1}x + a_{2,3} - a_{3,1}xu - a_{3,2}u^2\right)\partial_u \end{aligned}$$

is a general element of Lie algebra $\mathfrak{sl}_3(\mathbb{R})$, and

$$\alpha_2(X) = -3\,(a_{1,2} - a_{3,2}x)\,u_1 - 3\,a_{1,1} + 3\,a_{3,1}x$$

the corresponding 1-cocycle.

Also

$$X^{(5)}(P_5) = \alpha_5(X) \cdot P_5,$$

where

$$\alpha_5(X) = -6\,(a_{1,2} - a_{3,2}x)\,u_1 + 3\,a_{3,2}u - 9\,a_{1,1} + 9\,a_{3,1}x - 3\,a_{2,2}.$$

It is also easy to see that the function P_7 is a relative invariant.

Indeed,

$$X^{(7)}(P_7) = \alpha_7(X) \cdot P_7,$$

where

$$\alpha_7(X) = -32\,(a_{1,2} - a_{3,2}x)\,u_1 - 40\,a_{1,1} + 40\,a_{3,1}x - 8\,a_{2,2} + 8\,a_{3,2}u.$$

Cocycles α_2, α_5 and α_7 are not independent, and we have

$$16\alpha_2 + 8\alpha_5 - 3\alpha_7 = 0.$$

Another relative invariant we can get from the volume form $\Omega = dx \wedge du$ because

$$X(\Omega) = \alpha_0(X)\ \Omega,$$

where

$$\alpha_0(X) = 3\,a_{1,1} + 3\,a_{2,2} - 3\,a_{3,1}x - 3\,a_{3,2}u,$$

and

$$\alpha_0 - 2\alpha_2 + \alpha_5 = 0.$$

The last relative invariant we get from the contact form $\omega = du - u_1 dx$.

In this case

$$X^{(1)}(\omega) = \alpha_1(X)\,\omega,$$

where 1-cocycle α_1 has the following form

$$\alpha_1(X) = -(a_{1,2} - a_{3,2}x)u_1 + a_{1,1} + 2\,a_{2,2} - a_{3,1}x - 2\,a_{3,2}u,$$

and

$$2\alpha_0 - 3\alpha_1 + \alpha_2 = 0.$$

These relations between 1-cocycles allow us to construct the following invariant tensors.

Theorem 5.1. *The following tensors on jet spaces are $\mathbf{SL}_3(\mathbb{R})$-invariants:*

Function

$$Q_7 = \frac{P_7}{P_5^{8/3} P_2^{16/3}} \in C^\infty\left(\pi_{7,6}^{-1}(\Pi_{60})\right).$$

Differential 1-form

$$\omega_5 = \frac{P_5^{2/3}}{P_2^{5/3}}\omega \in \Omega^1(\Pi_{50}).$$

Differential 2-form

$$\Omega_5 = \frac{P_5}{P_2^2}\Omega \in \Omega^2(\Pi_{50}).$$

5.2 Algebra of projective differential invariants

Let's denote by τ_k and ν_k the vector bundles on $\mathbf{J}^k$ induced by projection $\pi_{k,1}$ from the canonical bundles τ_1, ν_1 on $\mathbf{J}^1$, where

$$\begin{aligned}\tau_1\left([L]_a^1\right) &= T_a L,\\ \nu_1\left([L]_a^1\right) &= T_a\mathbf{P}^2 \diagup T_a L.\end{aligned}$$

As we have seen symmetric differential forms

$$\Theta_2 = u_2\frac{dx^2}{2!}\otimes\overline{\partial}_u \in S^2\left(\tau_2^*\right)\otimes\nu_2$$

and

$$\Theta_5 = \left(u_5 - \frac{5u_3u_4}{u_2} + \frac{40}{9}\frac{u_3^3}{u_2^2}\right)\frac{dx^5}{5!}\otimes\overline{\partial}_u \in S^5\left(\tau_5^*\right)\otimes\nu_5$$

are $\mathbf{SL}_3(\mathbb{R})$-invariants.

Remark that all bundles τ_k and ν_k are 1-dimensional. Therefore, there exists a symmetric 3-form $\sigma \in S^3\left(\tau_5^*\right)$ such that

$$\Theta_5 = 60\ \sigma\cdot\Theta_2.$$

We call σ as *Study* 3-*form.*

This form is obviously $\mathbf{SL}_3(\mathbb{R})$-invariant and in affine coordinates can be written as follows

$$\sigma = \frac{P_5}{P_2}\ dx^3.$$

In addition to Study form we introduce a Study derivation as a such total derivation ∇ that

$$\sigma(\nabla,\nabla,\nabla) = 1.$$

Once more, in affine coordinates this derivation has the form

$$\nabla = \frac{P_2^{1/3}}{P_5^{1/3}}\ \frac{d}{dx}.$$

This is $\mathbf{SL}_3(\mathbb{R})$-invariant derivation.

It is easy to check that invariant Q_7 is an affine function in u_7 having the form

$$Q_7 = \frac{P_2^{2/3}}{P_5^{5/3}}u_7 + \cdots.$$

Applying the Study derivation we get an 8-th differential invariant

$$Q_8 = \nabla(Q_7) = \frac{P_2}{P_5^2}u_8 + \cdots, \tag{5.1}$$

and

$$Q_9 = \nabla(Q_8) = \frac{P_2^{\frac{4}{3}}}{P_5^{\frac{7}{3}}} u_9 + \cdots .$$

Continue in this way we get differential invariants in each order $k \geq 7$:

$$Q_k = \nabla^{k-7}(Q_7) = \frac{P_2^{\frac{k-5}{3}}}{P_5^{\frac{k-2}{3}}} u_k + \cdots$$

These relations show that differential invariants $Q_7, \ldots, Q_k$, separate $\mathbf{SL}_3(\mathbb{R})$-orbits in $\mathbf{J}^k$ if their projection to $\mathbf{J}^6$ coincides with regular orbit Π_{60}.

We call such orbits *regular.*

Let us specify now the notion of differential invariant for this $\mathbf{SL}_3(\mathbb{R})$-action.

First of all remark that all bundles $\pi_{k,k-1} : \mathbf{J}^k \to \mathbf{J}^{k-1}$ are affine, when $k \geq 2$. Therefore, we can talk about functions which are polynomial, or algebraic in derivatives $u_k, k \geq 2$.

We say that a function f defined in open and dense domain in manifold k-jets $\mathbf{J}^k$ is a $\mathbf{SL}_3(\mathbb{R})$-*differential invariant* (or simply *projective differential invariant*) of order k if

- $X^{(k)}(f) = 0$, for any vector field $X \in \mathfrak{sl}_3(\mathbb{R})$, and
- function f is a polynomial with respect to $u_\sigma, \sigma \geq 2$, and $P_2^{\pm 1/3}, P_5^{\pm 1/3}$.

Theorem 5.2. *1. Any projective differential invariant of order k is a polynomial of invariants $Q_7, ..., Q_k$.*

2. The algebra differential invariants separates regular orbits.

Proof. We'll use induction in k. Let Q be a differential invariant of order k which is a polynomial of degree n in u_k. Then, due to fact that Q_k is a polynomial of degree 1 in u_k, we can represent Q in the following way

$$Q = a_n Q_k^n + \cdots a_1 Q + a_0,$$

where a_i are functions on $(k-1)$-jets.

Applying vector fields $X \in \mathfrak{sl}_3(\mathbb{R})$ to both sides of this relation, we get

$$X^{(k-1)}(a_n) Q_k^n + \cdots + X^{(k-1)}(a_1) Q_k + X^{(k-1)}(a_0) = 0.$$

Therefore, $X^{(k-1)}(a_i) = 0$ for all $i = 0, ..., n, X \in \mathfrak{sl}_3(\mathbb{R})$, and we can use induction. □

6. Projective equivalence of plane curves

6.1 $\mathbf{SL}_3(\mathbb{R})$- action

Let L and $\widetilde{L}$ be smooth plane curves, and let $L^{(k)}, \widetilde{L}^{(k)} \subset \mathbf{J}^k$ be their prolongations. We say that L and $\widetilde{L}$ are *projectively equivalent* if $g(L) = \widetilde{L}$, for some element $g \in \mathbf{SL}_3(\mathbb{R})$.

Let's

$$Q_k(L) = Q_k|_{L^{(k)}}$$

be the value of invariant Q_k on the curve L.

Function $Q_7(L)$ is called *projective curvature of the curve.*

We will consider such curves L that function $Q_7(L)$ is a local coordinate on it. It is equivalent that function $\nabla(Q_7) = Q_8$ does not vanish on L.

To distinguish this situation we say that curve L is a *regular* at point $a \in L$, if its 5-jet belongs to the regular orbit, $[L]_a^5 \in \Pi_{50}$, and $Q_8(L)$ does not vanish at point $a \in L$.

Therefore, in a neighborhood of a regular point function $Q_7(L)$ is a local coordinate and

$$Q_8(L) = \Phi(Q_7(L))$$

for some smooth function Φ.

We call this function *defining function* of the curve.

We say that two plane curves L and $\widetilde{L}$ are projectively equivalent at points $a \in L$ and $\widetilde{a} \in \widetilde{L}$ if there exist a projective transformation ϕ such that $\phi(a) = \widetilde{a}$ and image $\phi(L)$ of curve L and $\widetilde{L}$ coincide in a neighborhood of point $\widetilde{a}$.

Theorem 6.1. *Two plane curves L and $\widetilde{L}$ are projectively equivalent in neighborhoods of regular points $a \in L$ and $\widetilde{a} \in \widetilde{L}$ if and only if $Q_7(L)(a) = Q_7\left(\widetilde{L}\right)(\widetilde{a})$ and their defining functions coincide in a neighborhood of point $Q_7(L)(a)$.*

Proof. The necessity condition is obvious. Let's prove the sufficiency, and let Φ be the defining function. Consider ordinary differential equation

$$Q_8 - \Phi(Q_7) = 0 \tag{6.1}$$

of the 8-th order.

Curves L and $\widetilde{L}$ are local solutions of this equation.

Relation (5.1) shows that solutions of the above differential equation are uniquely defined by their 8-jets.

Values of Q_7 and Q_8 on 8-jets $[L]_a^8$ and $[\widetilde{L}]_{\widetilde{a}}^8$ equal and therefore these jets belong to the regular orbit. Therefore,

$$\phi^{(8)}\left([L]_a^8\right) = [\widetilde{L}]_{\widetilde{a}}^8,$$

for some projective transformation ϕ.

Moreover, projective transformations are symmetries of differential equation (6.1). Hence, $\phi(L)$ is a solution (6.1) too. But 8-jets of $\widetilde{L}$ and $\phi(L)$ at point $\widetilde{a}$ equal. Therefore, due to uniqueness of solutions, $\widetilde{L} = \phi(L)$. □

Theorem 2 allows us to get normal forms of plane curves in a neighborhood of regular point.

Theorem 6.2. *Let L be a plane curve and let $a \in L$ be a regular point. Then there are affine coordinates (x, y) in a neighborhood $a \in L$, such that $x(a) = y(a) = 0$, and curve L is given by equation $y = Y(x)$, where*

$$\begin{aligned} Y(x) &= x^2 + \frac{2}{5}x^5 + \frac{108}{35}k_7x^7 + \left(\frac{1}{2} + \frac{81}{35}k_8\right)x^8 + \left(\frac{1944}{35}k_7^2 + \frac{54}{35}k_9\right)x^9 + \\ &+ \left(\frac{54}{5}k_7 + \frac{2916}{25}k_7k_8 + \frac{162}{175}k_{10}\right)x^{10} + \cdots, \end{aligned}$$

and k_i are values of $Q_i(L)$ at point a.

Remark 6.3. *If Φ is the defining function of the curve then coefficients k_i can be computed as follows:*

$$k_8 = \Phi(k_7), k_9 = \Phi'(k_7)\Phi(k_7), k_{10} = \Phi''(k_7)\Phi^2(k_7) + \Phi'(k_7)^2\Phi(k_7).$$

6.2 Cubics

As an example of application of the above theorem let's consider cubic curves. As we have seen these curves are solutions of equation

$$u_2P_7u_9 + P_8 = 0.$$

The left hand side of the equation is an obviously relative invariant.

This invariant can be written in terms of the known invariants as follows:

$$(P_2P_5)^5\left(-12600\,Q_7\,Q_9 + 14175\,Q_8^2 + 1225\,Q_8 - 259200\,Q_7^3 + \frac{343}{36}\right).$$

Therefore, if the cubic curve is an irreducible, i.e. is not union of quadric and straight lines, then this curve satisfies the 9-th order differential equation

$$-12600\,Q_7\,Q_9 + 14175\,Q_8^2 + 1225\,Q_8 - 259200\,Q_7^3 + \frac{343}{36} = 0. \tag{6.2}$$

The leading term of the last equation has the form

$$-12600\frac{Q_7P_2^{\frac{4}{3}}}{P_5^{\frac{7}{3}}}u_9 + \cdots.$$

Singularities for this equation in regular orbit belong to hyper surface

$$Q_7 = 0.$$

In other words, on cubic curves we have three types of singularities:

- *general singular points*: 5-jet of the curve belongs to a singular orbit,
- *non regular points:* invariant Q_8 vanishes,
- *projectively flat points*: projective curvature Q_7 vanishes.

Theorem 6.4. *Two plane connected cubic curves are projectively equivalent if and only if there are regular points on them where projective curvatures and their Study derivatives coincide.*

Proof. The proof of local projective equivalency is similar to proof of theorem 5. The rest follows from the fact that cubics are algebraic curves. □

Remark 6.5. *This result can be reformulated as follows. Let L be a plane curve. Define a new plane curve $\mathfrak{I}(L)$ as a image of the map $a \in L \longmapsto (Q_7(L)(a), Q_8(L)(a)) \in \mathfrak{I}(L)$, and let $\mathfrak{I}_0(L) \subset \mathfrak{I}(L)$ is the image of regular points. Then the above theorem claims, that two connected cubics L and $\widetilde{L}$ are projectively invariant if and only if $\mathfrak{I}_0(L) = \mathfrak{I}_0\left(\widetilde{L}\right)$.*

7. Special curves on projective plane

7.1 W-curves

Two classes of projective curve we get from description of singular orbits: straight lines and quadrics. The third class come from the description of singularities for cubics: these are curves of zero projective curvature, or solutions of differential equation of 7-th order:

$$Q_7 = 0.$$

This class is a subclass of so-called W-curves, introduced by Lie and Klein in ([4]).

Namely, straightforward computations show the following lemma is valid.

Lemma 7.1. $\mathbf{SL}_3(\mathbb{R})$*-orbits are transversal to curve $L^{(8)}$ at points where $Q_8 \neq 0$.*

Therefore, curves L for which $\mathbf{SL}_3(\mathbb{R})$-orbits of prolongations $L^{(8)}$ have dimension less then 9 should be solutions of the differential equation $Q_8 = 0$.

In other words, such curves have constant projective curvature. They are called W-curves (see, [3]).

It follows from the above lemma that trajectories of vector fields from our Lie algebra $\mathfrak{sl}_3$ are W-curves. Counting dimensions shows that the dimension of the space of solutions passing through a point $a \in \mathbf{P}^2$ for the differential equation $Q_8 = 0$ at regular point $a \in \mathbf{P}^2$ equals 7 that coincide with dimension of nonparametric trajectories of vector fields from $\mathfrak{sl}_3$.

The 3-rd description of W-curves was proposed by Klein and Lie. Namely, let's take three straight lines l_1, l_2 and l_3 on projective plane which are in general position and a plane curve L. Considering tangent line T_aL as a straight line on the plane, we define a number $j_L(a)$ to be equal the value of j-invariant for 4 points $[a, T_aL \cap l_1, T_aL \cap l_2, T_aL \cap l_3]$ on T_aL. Consider now such curves for which j-number j_L is a constant. They are also W-curves.

We have the following equivalent description of regular W-curves (see also [3]).

Theorem 7.2. *1. W-curves are solutions of 8-th order differential equation $Q_8 = 0$.*

2. W-curves are non parametrized trajectories of vector fields from the projective Lie algebra $\mathfrak{sl}_3$.

3. W-curves are curves of constant projective curvature.

4. If projective curvature $Q_7(L)$ of a curve L is a constant and equal $k \neq 0$, then the j-number of this curve equal $\frac{k^3}{k^3-\lambda}$, where $\lambda = \frac{675}{21\,952}$.

Corollary 7.3. *If a regular cubic is a W-curve then its projective curvature equals*

$$\frac{7\sqrt[3]{5}}{360},$$

and j-number equals

$$-\frac{7^6}{98\,297\,351}.$$

7.2 Study extremals

Another class of curves we obtain from the Study differential.

To this end let's consider the following functional

$$L \longmapsto \int_L \sqrt[3]{\sigma},$$

or in affine coordinates

$$y\left(x\right)\longmapsto\int_{a}^{b}\frac{\sqrt[3]{9\,y^{(5)}\left(y^{(2)}\right)^{2}+40\,\left(y^{(3)}\right)^{3}-45y^{(2)}y^{(3)}\,y^{(4)}}}{y^{(2)}}dx,$$

assuming that L is not a straight line, and call it *Study functional.*

Straightforward computations show that the following theorem is valid.

Theorem 7.4. *Extremals of the Study functional are solutions of the following differential equation of the* 10*-th order:*

$$\frac{P_5}{P_2^2}\left(Q_{10}-24Q_7Q_8\right)=0. \tag{7.1}$$

Corollary 7.5. *W-curves are extremals of the Study functional.*

Corollary 7.6. *If a regular cubic L is an extremal of the Study functional then L is the W-curve.*

8. Defining functions

In this section we consider a behavior of defining functions for cubics and Study extremals. It is worth to note that for straight lines, quadrics and W-curves the defining functions do not exist.

8.1 Cubics

Let Φ be the defining function of a cubic L considered in a neighborhood of regular point. Then, applying the Study derivative to the relation $Q_8=\Phi\left(Q_7\right)$, we get $Q_9=\Phi'\left(Q_7\right)\Phi\left(Q_7\right)$. Relation (6.2) can be rewritten now as a differential equation for defining function $\Phi\left(\tau\right)$:

$$\frac{343}{36}-259200\,\tau^3-12600\,\tau\Phi\Phi'+14175\,\Phi^2+1225\,\Phi=0.$$

Integrating this equation we get the following relation between invariants Q_7 and Q_8 which depends on arbitrary constant c and has the following form

$$F^3+cG{Q_7}^9=0,$$

where

$$\begin{aligned}F\ =\ &\frac{49}{147456}{Q_8}^4+\frac{343}{3317760}{Q_8}^3+\left(\frac{2401}{199065600}+\frac{7}{192}{Q_7}^3\right){Q_8}^2\\&+\left(-\frac{49}{25920}{Q_7}^3+\frac{16807}{26873856000}\right)Q_8\\&+\left({Q_7}^3-\frac{343}{1036800}\right)\left({Q_7}^3-\frac{343}{9331200}\right)\end{aligned} \tag{8.1}$$

and

$$G = 117649 - 6401203200\, {Q_7}^3 + 18151560\, Q_8 + 583443000\, {Q_8}^2 + \tag{8.2}$$
$$87071293440000\, {Q_7}^6 - 493807104000\, {Q_7}^3 Q_8$$
$$+3174474240000\, {Q_7}^3 {Q_8}^2 + 7001316000\, {Q_8}^3 + 28934010000\, {Q_8}^4.$$

In other words, regular cubics are projectively defined by constant c.

8.2 Study extremals

Rewriting Euler equation (7.1) in terms of defining function we get the following differential equation

$$\Phi^2\Phi'' + \Phi\Phi'^2 - 24\tau\Phi = 0.$$

Integrating, we get the following relation between invariants Q_7 and Q_8 :

$$Q_8^2 - \left(8Q_7^3 - c_1 Q_7 + c_2\right) = 0, \tag{8.3}$$

which depends on two arbitrary constant c_1 and c_2.

Summarizing, we arrive at the following description of non singular cubics and Study extremals.

Theorem 8.1.
- *Projective classes of regular cubics are defined by an arbitrary constant c, where*

  $$F^3 + cG{Q_7}^9 = 0,$$

  and expressions for invariants F and G are given in (8.1) and (8.2).
- *Projective classes of regular Study extremals are defined by relation (8.3) depending on two arbitrary constants c_1 and c_2.*

References

[1] J. L. Coolidge, "A treatise on algebraic plane curves", Dover Publ., NY, 1959

[2] G. H. Halphen, "Sur les invariants différentiels", Paris: Gauthier-Villars, 1878

[3] F. Klein, W. Blaschke, "Vorlesungen über höhere Geometrie", Berlin, J. Springer, 1926.

[4] F. Klein, S. Lie, “Uber diejenigen ebenen Curven, weiche aurch ein geschlossenes System von einfach unedlich veilen vertauschbaren linearen Transformationen in sich ubergehen”, Math. Annalen, Vol. 4, pp. 50-85, 1871.

[5] E.P. Lane, “Projective differential geometry of curves and surfaces”, The University of Chicago press, Chicago-Illinois, 1932

[6] A. Lascoux, “Differential equations of a plane curve”, Bull. Sc. Math., Vol. 130, pp. 354-359, 2006.

[7] E.J. Wilczynski, “Interpretation of the symplest integral invariant of projective geometry”, Proc. Nat. Acad. Sci. USA, Vol. 2, No. 4, pp. 248-52, 1916