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The existence of martingale solutions to the stochastic Boussinesq equations

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Abstract

A stochastic Boussinesq model for the Bénard problem is considered as a system of stochastic Navier-Stokes equations and transport equation in $\mathbb{R}^{d-1} \times [0,1]$, $d = 2,3$. The existence of a martingale solution is proved. The construction of the solution is based on the Faedo-Galerkin approximation and the compactness method.

Key words: Stochastic Boussinesq equations, martingale solution, compactness method.
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1. Introduction

We consider the Boussinesq model for the Bénard problem with random influences in $\mathbb{R}^{d-1} \times [0,1]$, where $d = 2,3$,

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u - \nu \Delta u - \vartheta e_d + \nabla p = f_1(t) + G_1(u,\vartheta)\, dW_1(t), \quad t \in [0,T] \quad (1.1)$$

$$\frac{\partial \vartheta}{\partial t} + (u \cdot \nabla)\vartheta - \kappa \Delta \vartheta - u_d = f_2(t) + G_2(u,\vartheta)\, dW_2(t), \quad (1.2)$$

with the incompressibility condition

$$\mathrm{div} u = 0, \tag{1.3}$$

with the boundary conditions in the vertical direction

$$u = 0 \quad \text{when} \quad x_d = 0 \quad \text{or} \quad x_d = 1, \tag{1.4}$$

$$\vartheta = 0 \quad \text{when} \quad x_d = 0 \quad \text{or} \quad x_d = 1. \tag{1.5}$$

and the periodic conditions in the horizontal directions:

$$p, u, \vartheta, \frac{\partial u}{\partial x_i}, \frac{\partial \vartheta}{\partial x_i} \quad \text{are periodic in the } x_i \text{ directions, } 1 \le i \le d-1. \tag{1.6}$$

The last condition means that for some $l > 0$ (when $d = 2$) or $l, L > 0$ (when $d = 3$)

$$\varphi_{|x_1=0} = \varphi_{|x_1=l}, \qquad \text{if } d = 2, 3,$$
$$\varphi_{|x_2=0} = \varphi_{|x_2=L}, \qquad \text{if } d = 3$$

for the corresponding functions φ.

The functions $u = u(t,x)$ and $p = p(t,x)$ are interpreted as the velocity and pressure of the fluid. Function $\vartheta = \vartheta(t,x)$ represents the temperature of the fluid (see [11]), here f_1, f_2 stand for the deterministic external forces and $G_1(u,\vartheta)\, dW_1(t)$, $G_2(u,\vartheta)\, dW_2(t)$, where W_1, W_2 are independent Wiener processes, are the random forces. This model has been studied by Foiaş, Manley and Temam [11] and Ghidaglia [12] in the deterministic case. The stochastic case is considered by Duan and Millet [8] and Ferrario [9] in 2D domains of the form $\mathbb{R} \times [0,1]$. In [9], the semigroup approach is used. The existence and uniqueness results and the existence of an invariant measure are proven for the model with an additive noise. In [8], large deviations are considered. In particular, the existence and uniqueness theorems in 2D case with a multiplicative noise term which may depend on the gradient of (u,ϑ) are proven. We generalize the existence result to the 3D case using a different approach which is independent of the method developed in [8].

The above problem can be written in an abstract form as the following initial-value problem in appropriate Hilbert space which is a Cartesian product of spaces used for the Navier-Stokes equations and spaces used in the theory of the transport equation, i.e.

$$d\phi + \big[A\phi + B(\phi) + R\phi\big]\, dt = f(t)\, dt + G(\phi)\, dW(t), \qquad t \in [0,T],$$

with the initial condition

$$\phi(0) = \phi_0,$$

where $W(t) = (W_1(t), W_2(t))$, $f = \big(f_1(t), f_2(t)\big)$, $\phi = (u, \vartheta)$ and $\phi_0 = (u_0, \vartheta_0)$. Here, A and R are linear operators and B is a bilinear mapping. We impose rather general conditions on the noise $G(\phi)dW(t) = G_1(u,\vartheta)\,dW_1(t) + G_2(u,\vartheta)\,dW_2(t)$, see Section 3 for details. These assumptions cover the following special case

$$G(\phi(t))\,dW(t) := \sum_{i=1}^{\infty} \big[\big(b^{(i)}(x)\cdot\nabla\big)\phi(t,x) + c^{(i)}(x)\phi(t,x)\big]\,d\beta^{(i)}(t),$$

where $\{\beta^{(i)}\}_{i\in\mathbb{N}}$ are independent standard Brownian motions, see Section 5.

We prove the existence of a martingale solution. The construction of the solution is based on the Faedo-Galerkin approximation, i.e.

$$\begin{cases} d\phi_n(t) = -\big[P_n A\phi_n(t) + B_n\big(\phi_n\big) + P_n R(\phi_n(t)) - P_n f(t)\big]\,dt \\ \qquad\qquad\quad +P_n G\big(\phi_n\big)\,dW(t), \quad t\in[0,T], \\ \phi_n(0) = P_n\phi_0. \end{cases}$$

The solutions ϕ_n to the Galerkin equations generate a sequence of laws $\{\mathcal{L}(\phi_n);\ n\in\mathbb{N}\}$ on appropriate functional spaces. To prove that this sequence of probability measures is weakly compact we need appropriate tightness criteria.

Our approach is closely related to the method used by Flandoli and Gątarek [10] to the stochastic Navier-Stokes equations in a bounded domain, where also the Faedo-Galerkin approximation is used. In [10] the tightness of an appropriate sequence of measures is proven by means of some compactness results in fractional Sobolev spaces. In the present paper we prove the tightness of $\{\mathcal{L}(\phi_n);\ n\in\mathbb{N}\}$ using a different criterion of the compactness than used in [10].

Thus we first concentrate on another tightness criterion which we formulate in an abstract setting. More precisely, let $\mathbb{H}$ and $\mathbb{V}$ be two real separable Hilbert spaces such that $\mathbb{V}\hookrightarrow\mathbb{H}$, the embedding being dense and compact. Moreover, we assume that there exists a third real separable Hilbert space such that the embedding $U\hookrightarrow\mathbb{V}$ is dense and continuous. Using the classical Dubinsky Theorem, [23] and some ideas of Mikulevicius and Rozovskii [17], we prove a certain criterion for relative compactness of a set $\mathcal{K}$ in the intersection space

$$\mathcal{Z} := \mathcal{C}([0,T];U')\cap L^q_w(0,T;\mathbb{V})\cap L^q(0,T;\mathbb{H})\cap\mathcal{C}([0,T];\mathbb{H}_w).$$

To be precise we show that the following three conditions, see Lemma 2.3,

(a) $\sup_{u\in\mathcal{K}}\sup_{s\in[0,T]}|u(s)|_{\mathbb{H}}<\infty$,

(b) $\sup_{u\in\mathcal{K}} \int_0^T \|u(s)\|_{\mathbb{V}}^q \, ds < \infty$,

(c) $\lim_{\delta\to 0} \sup_{u\in\mathcal{K}} \sup_{\substack{s,t\in[0,T] \\ |t-s|\le\delta}} |u(t)-u(s)|_{U'} = 0$,

are sufficient for the relative compactness of $\mathcal{K}$ in $\mathcal{Z}$. Let us notice that assumption (c) is, in fact, the assumption on the modulus of continuity.

Mikulevicius and Rozovskii [17] proved the existence of a martingale solution of the stochastic Navier-Stokes equations in $\mathbb{R}^d$, $d \ge 2$. Note that in this case the embedding $H^1(\mathbb{R}^d) \hookrightarrow L^2(\mathbb{R}^d)$ is not compact. In their approach the space $L^2(\mathbb{R}^d)$ is compactly embedded in the Fréchet space $H_{loc}^{-k_0}(\mathbb{R}^d)$ for sufficiently large k_0. Then, they proved a compactness criterion in the intersection space

$$\mathcal{C}([0,T]; H_{loc}^{-k_0}(\mathbb{R}^d)) \cap \mathcal{C}([0,T]; L_w^2(\mathbb{R}^d)) \cap L_w^2(0,T; H^1(\mathbb{R}^d)) \cap L^2(0,T; L_{loc}^2(\mathbb{R}^d)).$$

(The letter w indicates the weak topology.) The main difference between our paper and [17] is that we formulate this criterion in an abstract Hilbert spaces setting. However, since we assume compactness of the embedding $\mathbb{V} \hookrightarrow \mathbb{H}$, in the proof Lemma 2.3 we can use the Dubinsky Theorem.

Using the deterministic compactness criterion formulated in Lemma 2.3 and the Aldous condition in the form given by Métivier [15], we find a certain tightness criterion for the laws on the space $\mathcal{Z}$, see Corollary 2.6. Assumptions in this tightness criterion are expressed in terms of uniform estimates on expected values of the norms in (a) and (b). The assumption corresponding to the modulus of continuity is given by the Aldous condition, see Métivier [15] and Section 2.

Furthermore, the construction of a martingale solution differs from the approach by Mikulevicius and Rozovskii. We apply the method used by Da Prato and Zabczyk in [7], Section 8. This method is based on the Skorokhod Theorem and the martingale representation Theorem. This is also the method used in [10].

The paper is organized as follows. In Section 2 we are concerned with the compactness result. In Section 3, we formulate the Boussinesq problem as an abstract stochastic evolution equation in appropriate Sobolev spaces. The main theorem concerning existence and construction of the martingale solutions is in Section 4. Some auxilliary results connected with the proof are given in Appendices A and B. In Section 5, we consider an example of the noise.

2. Compactness result

Using the classical Dubinsky Theorem, see [23], we will prove a certain compactness criterion analogous to that contained in Lemma 2.7 in [17]. However, we put it into the abstract framework in Hilbert spaces. Using the Aldous condition in the form given by Métivier, [15], we obtain a certain tightness criterion.

2.1 Compactness criterion

Let $(\mathbb{H},(\cdot|\cdot))$ and $\big(\mathbb{V},\big((\cdot|\cdot)\big)\big)$ be abstract real separable Hilbert spaces such that the embedding $\mathbb{V}\hookrightarrow\mathbb{H}$ is dense and compact. Identifying $\mathbb{H}$ with its dual $\mathbb{H}'$, we have the following embeddings

$$\mathbb{V}\hookrightarrow\mathbb{H}\cong\mathbb{H}'\hookrightarrow\mathbb{V}'.$$

Assume that there exists a real separable Hilbert space U such that $U\hookrightarrow\mathbb{V}$, the embedding being dense and continuous. Thus we have

$$U\hookrightarrow\mathbb{V}\hookrightarrow\mathbb{H}\cong\mathbb{H}'\hookrightarrow\mathbb{V}'\hookrightarrow U'. \tag{2.1}$$

Then in particular, $\mathbb{H}\hookrightarrow U'$ is compact.

Let $q\in(1,\infty)$. Let us consider the following three functional spaces

$$\begin{aligned}
\mathcal{C}([0,T],U') &:= \text{the space of continuous functions } u:[0,T]\to U' \text{ with the to-}\\
&\quad\ \text{pology } \mathcal{T}_1 \text{ induced by the norm } \|u\|_{\mathcal{C}([0,T];U')}:=\sup_{t\in[0,T]}\|u(t)\|_{U'},\\
L^q_w(0,T;\mathbb{V}) &:= \text{the space } L^q(0,T;\mathbb{V}) \text{ with the weak topology } \mathcal{T}_2,\\
L^q(0,T;\mathbb{H}) &:= \text{the space } L^q(0,T;\mathbb{H}) \text{ with the topology } \mathcal{T}_3 \text{ induced}\\
&\quad\ \text{by the norm.}
\end{aligned}$$

Let $\mathbb{H}_w$ denote the Hilbert space $\mathbb{H}$ endowed with the weak topology. Consider the ball

$$\mathbb{B}:=\{x\in\mathbb{H};\ \ |x|_{\mathbb{H}}\le r\}.$$

It is well-known that the weak topology induced on $\mathbb{B}$ is metrizable, see [4]. Let q denote the metric compatible with the weak topology on $\mathbb{B}$. Let us consider the following space

$$\begin{aligned}
\mathcal{C}([0,T];\mathbb{B}_w) &= \text{the space of weakly continuous functions } u:[0,T]\to\mathbb{H}\\
&\quad\ \text{and such that } \sup_{t\in[0,T]}|u(t)|_{\mathbb{H}}\le r.
\end{aligned} \tag{2.2}$$

The space $\mathcal{C}([0,T];\mathbb{B}_w)$ is metrizable with

$$\varrho(u,v)=\sup_{t\in[0,T]}q(u(t),v(t)). \tag{2.3}$$

Since by the Banach-Alaoglu Theorem $\mathbb{B}_w$ is compact, $(\mathcal{C}([0,T];\mathbb{B}_w),\varrho)$ is a complete metric space. Moreover, $u_n \to u$ in $\mathcal{C}([0,T];\mathbb{B}_w)$ iff for all $h \in \mathbb{H}$:

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \left|(u_n(t)-u(t)|h)_{\mathbb{H}}\right| = 0.$$

The following lemma says that any sequence $(u_n) \subset \mathcal{C}([0,T];\mathbb{B})$ convergent in $\mathcal{C}([0,T];U')$ is also convergent in the space $\mathcal{C}([0,T];\mathbb{B}_w)$. It is closely related to the lemma due to Strauss, see [19], that says:

$$L^\infty(0,T;\mathbb{H}) \cap \mathcal{C}([0,T];U'_w) \subset \mathcal{C}([0,T];\mathbb{H}_w), \tag{2.4}$$

where $\mathcal{C}([0,T];U'_w)$ denotes the space of U'-valued weakly continuous functions.

Lemma 2.1. *Let $u_n : [0,T] \to \mathbb{H}$, $n \in \mathbb{N}$ be functions such that*

(i) $\sup_{n\in\mathbb{N}} \sup_{s\in[0,T]} |u_n(s)|_{\mathbb{H}} \le r$,

(ii) $u_n \to u$ *in* $\mathcal{C}([0,T];U')$.

Then $u, u_n \in \mathcal{C}([0,T];\mathbb{B}_w)$ and $u_n \to u$ in $\mathcal{C}([0,T];\mathbb{B}_w)$ as $n \to \infty$.

Proof. By (2.4) we infer that $u_n \in \mathcal{C}([0,T];\mathbb{H}_w)$. We claim that

$$u_n \to u \quad \text{in} \quad \mathcal{C}([0,T];\mathbb{B}_w) \quad \text{as} \quad n \to \infty,$$

i.e. that for all $h \in H$

$$\lim_{n\to\infty} \sup_{s\in[0,T]} \left|(u_n(s)-u(s)|h)_{\mathbb{H}}\right| = 0. \tag{2.5}$$

Indeed, first let us fix $h \in U$. Then for all $s \in [0,T]$ we have

$$\left|(u_n(s)-u(s)|h)_{\mathbb{H}}\right| = \left|\big(u_n(s)-u(s)|h\big)\right| \le |u_n(s)-u(s)|_{U'} \cdot \|h\|_U.$$

Since $u_n \to u$ in $\mathcal{C}([0,T];U')$,

$$\sup_{s\in[0,T]} \left|(u_n(s)-u(s)|h)_{\mathbb{H}}\right| \le \sup_{s\in[0,T]} |u_n(s)-u(s)|_{U'} \cdot \|h\|_U \to 0$$

as $n \to \infty$.

To show that condition (2.5) holds for all $h \in \mathbb{H}$ let us fix $h \in \mathbb{H}$ and $\varepsilon > 0$. Since U is dense in $\mathbb{H}$, there exists $h_\varepsilon \in U$ such that $|h-h_\varepsilon|_{\mathbb{H}} \le \varepsilon$. Using (i), we infer that for all $s \in [0,T]$ the following estimates hold

$$\begin{aligned}
\left|(u_n(s)-u(s)|h)_{\mathbb{H}}\right| &\le \left|(u_n(s)-u(s)|h-h_\varepsilon)_{\mathbb{H}}\right| + \left|(u_n(s)-u(s)|h_\varepsilon)_{\mathbb{H}}\right| \\
&\le |u_n(s)-u(s)|_{\mathbb{H}} |h-h_\varepsilon|_{\mathbb{H}} + \left|(u_n(s)-u(s)|h_\varepsilon)_{\mathbb{H}}\right| \\
&\le \varepsilon \cdot \|u_n - u\|_{L^\infty(0,T;\mathbb{H})} + \left|(u_n(s)-u(s)|h_\varepsilon)_{\mathbb{H}}\right| \\
&\le 2\varepsilon \cdot \sup_{n\in\mathbb{N}} \|u_n\|_{L^\infty(0,T;\mathbb{H})} + \left|(u_n(s)-u(s)|h_\varepsilon)_{\mathbb{H}}\right| \\
&\le 2\varepsilon r + \sup_{s\in[0,T]} \left|(u_n(s)-u(s)|h_\varepsilon)_{\mathbb{H}}\right|.
\end{aligned}$$

Thus

$$\sup_{s\in[0,T]} \left|(u_n(s)-u(s)|h)_{\mathbb{H}}\right| \le 2\varepsilon r + \sup_{s\in[0,T]} \left|(u_n(s)-u(s)|h_\varepsilon)_{\mathbb{H}}\right|.$$

Passing to the upper limit as $n\to\infty$, we obtain

$$\limsup_{n\to\infty} \sup_{s\in[0,T]} \left|(u_n(s)-u(s)|h)_{\mathbb{H}}\right| \le 2r\varepsilon.$$

Since ε is arbitrary,

$$\lim_{n\to\infty} \sup_{s\in[0,T]} \left|(u_n(s)-u(s)|h)_{\mathbb{H}}\right| = 0.$$

Since $\mathcal{C}([0,T];\mathbb{B}_w)$ is a complete metric space, we infer that $u\in\mathcal{C}([0,T];\mathbb{B}_w)$ as well. The proof is thus complete. □

Let us recall the classical compactness criterion due to Dubinsky, [23], Theorem IV.4.1, see also [14].

Theorem 2.2. (Dubinsky) *Let E_0, E and E_1 be reflexive Banach spaces such that the embeddings $E_0 \hookrightarrow E \hookrightarrow E_1$ are continuous and the embedding $E_0 \hookrightarrow E$ is compact. Let $q\in(1,\infty)$ and let $\mathcal{K}$ be a bounded set in $L^q(0,T;E_0)$ consisting of functions equicontinuous in $\mathcal{C}([0,T];E_1)$. Then $\mathcal{K}$ is relatively compact in $L^q(0,T;E)$ and $\mathcal{C}([0,T];E_1)$.*

Using Theorem 2.2 and Lemma 2.1, we obtain compactness criterion analogous to the result due to Mikulevicius and Rozovskii, see [17].

Lemma 2.3. (see Lemma 2.7 in [17]) *Let $q\in(1,\infty)$ and let*

$$\mathcal{Z} := \mathcal{C}([0,T];U') \cap L^q_w(0,T;\mathbb{V}) \cap L^q(0,T;\mathbb{H}) \cap \mathcal{C}([0,T];\mathbb{H}_w) \tag{2.6}$$

and let $\mathcal{T}$ be the supremum of the corresponding topologies. Then a set $\mathcal{K}\subset\mathcal{Z}$ is $\mathcal{T}$-relatively compact if the following three conditions hold

(a) *for all $u\in\mathcal{K}$ and for all $t\in[0,T]$, $u(t)\in\mathbb{H}$ and*
$\sup_{u\in\mathcal{K}} \sup_{s\in[0,T]} |u(s)|_{\mathbb{H}} < \infty$,

(b) $\sup_{u\in\mathcal{K}} \int_0^T \|u(s)\|^q_{\mathbb{V}}\, ds < \infty$, *i.e. $\mathcal{K}$ is bounded in $L^q(0,T;\mathbb{V})$,*

(c) $\lim_{\delta\to 0} \sup_{u\in\mathcal{K}} \sup_{\substack{s,t\in[0,T]\\ |t-s|\le\delta}} |u(t)-u(s)|_{U'} = 0$.

Proof. Without loss of generality we can assume that $\mathcal{K}$ is a closed subset of $\mathcal{Z}$. Because of the assumption (a) we may consider the metric subspace $\mathcal{C}([0,T];\mathbb{B}_w) \subset \mathcal{C}([0,T];\mathbb{H}_w)$ defined by (2.2) and (2.3) with $r := \sup_{u\in\mathcal{K}} \sup_{s\in[0,T]} |u(s)|_{\mathbb{H}}$. By assumption (b) the restriction to the set $\mathcal{K}$ of the weak topology in $L^q_w(0,T;\mathbb{V})$ is metrizable. Since the restrictions to $\mathcal{K}$ of the four topologies considered in $\mathcal{Z}$ are metrizable, compactness of a subset of $\mathcal{Z}$ is equivalent to its sequential compactness.

Let (u_n) be a sequence in $\mathcal{K}$. By the Banach-Alaoglu Theorem condition (b) yields that $\mathcal{K}$ is compact in $L^q_w(0,T;\mathbb{V})$. Condition (c) implies that the functions (u_n) are equicontinuous. By Theorem 2.2 assumptions (b) and (c) imply that $\mathcal{K}$ is compact in $\mathcal{C}([0,T];U') \cap L^q(0,T;\mathbb{H})$. Hence in particular, there exists a subsequence, still denoted by (u_n), convergent in $\mathcal{C}([0,T];U')$. Therefore by Lemma 2.1 (u_n) is convergent in $\mathcal{C}([0,T];\mathbb{B}_w)$. This completes the proof of the statement. □

2.2 The Aldous condition and tightness

Let $(\mathbb{S},\varrho)$ be a separable and complete metric space.

Definition 1. Let $u \in \mathcal{C}([0,T],\mathbb{S})$. The modulus of continuity of u on $[0,T]$ is defined by

$$\mathrm{m}(u,\delta) := \sup_{s,t\in[0,T],|t-s|\le\delta} \varrho(u(t),u(s)), \qquad \delta > 0.$$

Let $(\Omega,\mathcal{F},\mathbb{P},)$ be a probability space with filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions, see [16], and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of continuous $\mathbb{F}$-adapted and $\mathbb{S}$-valued processes.

Definition 2. We say that a sequence (X_n) of $\mathbb{S}$-valued random variables satifies condition $[\tilde{\mathbf{T}}]$ iff

$[\tilde{\mathbf{T}}]$ $\quad \forall\,\varepsilon > 0 \quad \forall\,\eta > 0 \quad \exists\,\delta > 0$:

$$\sup_{n\in\mathbb{N}} \mathbb{P}\big\{\mathrm{m}(X_n,\delta) > \eta\big\} \le \varepsilon. \tag{2.7}$$

Remark. Let $\mathbb{P}_n$ denote the law of X_n on $\mathcal{C}([0,T],\mathbb{S})$. For fixed $\eta > 0$ and $\delta > 0$ we denote

$$C_{\eta,\delta} := \{u \in \mathcal{C}([0,T],\mathbb{S}) : \ \mathrm{m}(u,\delta) \ge \eta\}.$$

Then condition

$$\mathbb{P}\big\{\mathrm{m}(X_n,\delta) > \eta\big\} \le \varepsilon$$

is equivalent to the following one

$$\mathbb{P}_n(C_{\eta,\delta}) \le \varepsilon.$$

Lemma 2.4. *Assume that (X_n) satisfies condition $[\tilde{\mathbf{T}}]$. Let $\mathbb{P}_n$ be the law of X_n on $\mathcal{C}([0,T],\mathbb{S})$, $n \in \mathbb{N}$. Then for every $\varepsilon > 0$ there exists a subset $A_\varepsilon \subset \mathcal{C}([0,T],\mathbb{S})$ such that*

$$\sup_{n\in\mathbb{N}} \mathbb{P}_n(A_\varepsilon) \geq 1 - \varepsilon$$

and

$$\lim_{\delta\to 0} \sup_{u\in A_\varepsilon} m(u,\delta) = 0. \tag{2.8}$$

Proof. Let us fix $\varepsilon > 0$. By condition $[\tilde{\mathbf{T}}]$ for each $k \in \mathbb{N}$ there exists $\delta_k > 0$ such that

$$\sup_{n\in\mathbb{N}} \mathbb{P}\big\{\mathrm{m}(X_n,\delta_k) > \frac{1}{k}\big\} \leq \frac{\varepsilon}{2^{k+1}}.$$

Then

$$\sup_{n\in\mathbb{N}} \mathbb{P}\big\{\mathrm{m}(X_n,\delta_k) \leq \frac{1}{k}\big\} \geq 1 - \frac{\varepsilon}{2^{k+1}}$$

or equivalently

$$\sup_{n\in\mathbb{N}} \mathbb{P}_n\big\{u \in \mathcal{C}([0,T],\mathbb{S}) : \ \mathrm{m}(u,\delta_k) \leq \frac{1}{k}\big\} \geq 1 - \frac{\varepsilon}{2^{k+1}}.$$

Let $B_k := \big\{u \in \mathcal{C}([0,T],\mathbb{S}) : \ \mathrm{m}(u,\delta_k) \leq \frac{1}{k}\big\}$ and let $A_\varepsilon := \bigcap_{k=1}^\infty B_k$. We assert that for each $n \in \mathbb{N}$ one has

$$\mathbb{P}_n\big(A_\varepsilon\big) \geq 1 - \varepsilon.$$

Indeed, we have the following estimates

$$\begin{aligned}
\mathbb{P}_n\big(\mathcal{C}([0,T],\mathbb{S}) \setminus A_\varepsilon\big) &\leq \mathbb{P}_n\big(\mathcal{C}([0,T],\mathbb{S}) \setminus \bigcap_{k=1}^\infty B_k\big) = \mathbb{P}_n\big(\bigcup_{k=1}^\infty \big(\mathcal{C}([0,T],\mathbb{S}) \setminus B_k\big)\big) \\
&\leq \sum_{k=1}^\infty \mathbb{P}_n\big(\mathcal{C}([0,T],\mathbb{S}) \setminus B_k\big) \leq \sum_{k=1}^\infty \frac{\varepsilon}{2^{k+1}} = \varepsilon.
\end{aligned}$$

Thus $\mathbb{P}_n(A_\varepsilon) \geq 1 - \varepsilon$.

To prove (2.8), let us fix $\tilde{\varepsilon} > 0$. Directly from the definition of A_ε, we infer that $\sup_{u\in A_\varepsilon} \mathrm{m}(u,\delta_k) \leq \frac{1}{k}$ for each $k \in \mathbb{N}$. Choose $k_0 \in \mathbb{N}$ such that $\frac{1}{k_0} \leq \tilde{\varepsilon}$ and let $\delta_0 := \delta_{k_0}$. Then for every $\delta \leq \delta_0$ we obtain the following estimate

$$\mathrm{m}(u,\delta) \leq \mathrm{m}(u,\delta_{k_0}) \leq \tilde{\varepsilon},$$

which completes the proof of (2.8) and of the Lemma. □

Now, we recall the Aldous condition which is connected with condition $[\tilde{\mathbf{T}}]$ (see [15] and [2]). This condition allows to investigate the modulus of continuity for the sequence of stochastic processes by means of stopped processes.

Definition 3. A sequence $(X_n)_{n\in\mathbb{N}}$ satisfies condition **[A]** iff

[A] $\forall\,\varepsilon > 0 \quad \forall\,\eta > 0 \quad \exists\,\delta > 0$ such that for every sequence $(\tau_n)_{n\in\mathbb{N}}$ of $\mathbb{F}$-stopping times with $\tau_n \le T$ one has

$$\sup_{n\in\mathbb{N}} \sup_{0\le\theta\le\delta} \mathbb{P}\big\{ \varrho\big(X_n(\tau_n+\theta), X_n(\tau_n)\big) \ge \eta \big\} \le \varepsilon.$$

Lemma 2.5. *(See [15], Th. 3.2 p. 29) Conditions* $[\mathbf{A}]$ *and* $[\tilde{\mathbf{T}}]$ *are equivalent* .

Using the compactness criterion formulated in Lemma 2.3 we obtain the following corollary which we will use to prove tightness of the laws defined by the Galerkin approximations.

Let us recall that $\mathbb{H}, \mathbb{V}, U$ are separable Hilbert spaces such that

$$U \hookrightarrow \mathbb{V} \hookrightarrow \mathbb{H},$$

where the embedding $\mathbb{V} \hookrightarrow \mathbb{H}$ is dense and compact and the embedding $U \hookrightarrow \mathbb{V}$ is dense and continuous.

Corollary 2.6. (tightness criterion) *Let $q \in (1,\infty)$ and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of continuous $\mathbb{F}$-adapted U'-valued processes such that*

(a) *there exists a positive constant C_1 such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\big[\sup_{s\in[0,T]} |X_n(s)|_{\mathbb{H}}\big] \le C_1,$$

(b) *there exists a positive constant C_2 such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\left[\int_0^T \|X_n(s)\|_{\mathbb{V}}^q\, ds\right] \le C_2,$$

(c) *$(X_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition* **[A]** in U'.

Let $\tilde{\mathbb{P}}_n$ be the law of X_n on $\mathcal{Z}$. Then for every $\varepsilon > 0$ there exist a compact subset K_ε of $\mathcal{Z}$ such that

$$\tilde{\mathbb{P}}_n(K_\varepsilon) \ge 1-\varepsilon.$$

Proof. Let $\varepsilon > 0$. By the Chebyshev inequality and (a) we infer that for any $r > 0$

$$\mathbb{P}\Big(\sup_{s\in[0,T]} |X_n(s)|_{\mathbb{H}} > r\Big) \le \frac{\mathbb{E}\big[\sup_{s\in[0,T]} |X_n(s)|_{\mathbb{H}}\big]}{r} \le \frac{C_1}{r}.$$

Let R_1 be such that $\frac{C_1}{R_1} \le \frac{\varepsilon}{3}$. Then

$$\mathbb{P}\Big(\sup_{s\in[0,T]} |X_n(s)|_{\mathbb{H}} > R_1\Big) \le \frac{\varepsilon}{3}$$

Let $B_1 := \big\{u \in \mathcal{Z} :\ \sup_{s\in[0,T]} |u(s)|_{\mathbb{H}} \le R_1\big\}$.

By the Chebyshev inequality and (b) we infer that for any $r > 0$

$$\mathbb{P}\big(\|X_n\|_{L^q(0,T;\mathbb{V})} > r\big) \le \frac{\mathbb{E}\big[\|X_n\|^q_{L^q(0,T;\mathbb{V})}\big]}{r^q} \le \frac{C_q}{r^q}.$$

Let R_2 be such that $\frac{C_q}{R_2^q} \le \frac{\varepsilon}{3}$. Then

$$\mathbb{P}\big(\|X_n\|_{L^q(0,T;\mathbb{V})} > R_2\big) \le \frac{\varepsilon}{3}.$$

Let $B_2 := \big\{u \in \mathcal{Z} :\ \|u\|_{L^q(0,T;\mathbb{V})} \le R_2\big\}$.

By Lemmas 2.5 and 2.4 there exists a subset $A_{\frac{\varepsilon}{3}} \subset \mathcal{C}([0,T], U')$ such that $\tilde{\mathbb{P}}_n\big(A_{\frac{\varepsilon}{3}}\big) \ge 1 - \frac{\varepsilon}{3}$ and

$$\lim_{\delta\to 0} \sup_{u\in A_{\frac{\varepsilon}{3}}} \sup_{\substack{s,t\in[0,T]\\ |t-s|\le\delta}} |u(t) - u(s)|_{U'} = 0.$$

It is sufficient to define K_ε as the closure of the set $B_1 \cap B_2 \cap A_{\frac{\varepsilon}{3}}$ in $\mathcal{Z}$. Since by Lemma 2.3 K_ε is compact in $\mathcal{Z}$, the proof is thus complete. □

3. Functional setting

Let $D = (0,l) \times (0,1)$ if $d = 2$ or $D = (0,l) \times (0,L) \times (0,1)$ if $d = 3$. Consider Hilbert spaces

$$\begin{aligned} V_1 = \{v \in H^1(D, \mathbb{R}^d) :\ & \mathrm{div} v = 0,\ \ v_{|x_d=0} = v_{|x_d=1} = 0, \\ & \text{and } v \text{ is periodic in the direction } x_1 \text{ with period } l \text{ if } d = 2,3 \\ & \text{and in direction } x_2 \text{ with period } L \text{ if } d = 3\} \end{aligned}$$

with the scalar product $\big((u|v)\big) = \int_D \nabla u(x) \cdot \nabla v(x)\, dx, \quad u, v \in V_1$ and

$$V_2 = \{f \in H^1(D, \mathbb{R}) : \ f_{|x_d=0} = f_{|x_d=1} = 0, \text{ and } f \text{ is periodic in the direction } x_1 \text{ with period } l \text{ if } d = 2, 3 \text{ and in direction } x_2 \text{ with period } L \text{ if } d = 3\}$$

with the scalar product $\big((f|g)\big) = \int_D \nabla f(x) \cdot \nabla g(x)\, dx$, $f, g \in V_2$. The boundary conditions are understood in the sense of traces; see [22], Section 2. Let $V = V_1 \times V_2$ be the product Hilbert space with the scalar product

$$\big((\phi|\psi)\big) := \int_D \nabla\phi(x) \cdot \nabla\psi(x)\, dx, \qquad \phi, \psi \in V$$

and the norm

$$\|\phi\|_V^2 := \big((\phi|\phi)\big) = \|u\|_{V_1}^2 + \|f\|_{V_2}^2, \qquad \phi = (u, f) \in V.$$

Let $H = H_1 \times H_2$, where

$$\begin{aligned} H_1 = \{u \in L^2(D, \mathbb{R}^d) : \ & \mathrm{div} u = 0, \text{ and } u \text{ is periodic in the direction } x_1 \\ & \text{with period } l \text{ if } d = 2, 3 \text{ and in direction } x_2 \text{ with period } L \text{ if } d = 3\}, \\ H_2 = \{g \in L^2(D, \mathbb{R}) : \ & \text{and } g \text{ is periodic in the direction } x_1 \text{ with period } l \\ & \text{if } d = 2, 3 \text{ and in direction } x_2 \text{ with period } L \text{ if } d = 3\}. \end{aligned}$$

The incompressibility condition and the periodic conditions present in the definition of the space H_1 are to be understood in the distributional sense. It is well known that H is a Hilbert space with the scalar product $\big(\phi|\psi\big) = \int_D \phi(x) \cdot \psi(x)\, dx$ and the norm $|\phi|_H^2 = \big(\phi|\phi\big)$, where $\phi, \psi \in H$. Moreover, the embedding $V \hookrightarrow H$ is well defined, compact and dense.

With the scalar products in V_1 and V_2 we can associate unbounded linear operators $A_i : H_i \supset D(A_i) \to H_i$, $i = 1, 2$ with domains $D(A_1) = V_1 \cap H^2(D, \mathbb{R}^d)$ and $D(A_2) = V_2 \cap H^2(D, \mathbb{R})$ by the following formulae

$$\begin{aligned} \big(A_1 u|v\big) &= \big((u|v)\big), \qquad u, v \in D(A_1), \\ \big(A_2 \vartheta|\theta\big) &= \big((\vartheta|\theta)\big), \qquad \vartheta, \theta \in D(A_2). \end{aligned}$$

Operators A_1 and A_2 are selfadjoint, positive with compact inverses. Let $D(A) := D(A_1) \times D(A_2)$ and

$$A = \nu A_1 \times \kappa A_2,$$

i.e. $A\phi = (\nu A_1 u, \kappa A_2 \vartheta)$, $\phi = (u, \vartheta) \in D(A)$.

Moreover, we can define the fractional power operators A^α, $\alpha \in \mathbb{R}$. The domains $D(A^\alpha)$ of these operators correspond to the Sobolev spaces $H^{2\alpha}(D, \mathbb{R}^{d+1})$

equipped with the suitable boundary conditions. In particular, $V = D(A^{\frac{1}{2}})$. The embedding of $D(A^{\alpha})$ into $D(A^{\alpha-\varepsilon})$ is compact for every $\alpha \in \mathbb{R}$ and $\varepsilon > 0$, see [21], Chapter II. We will also use the notation $V_\alpha := D(A^{\frac{\alpha}{2}})$.

Let us consider the following three-linear form

$$b_1(u, w, v) = \int_D (u \cdot \nabla w) v \, dx. \tag{3.1}$$

Using the Hőlder inequality and the Sobolev embedding Theorem, it is easy to prove the following inequalities

$$\begin{aligned} |b_1(u, w, v)| &\le |u|_{\mathbb{L}^4} \|v\|_{V_1} |v|_{\mathbb{L}^4} \\ &\le c\|u\|_{V_1} \|w\|_{V_1} \|v\|_{V_1}, \qquad u, w, v \in V_1 \end{aligned} \tag{3.2}$$

for some constant $c > 0$. Thus the form b_1 is continuous on V_1, see [22], Section 2. Moreover, if we define a bilinear map B_1 by $B_1(u, w) := b_1(u, w, \cdot)$, then by inequality (3.2) we infer that $B_1(u, w) \in V_1'$ for all $u, w \in V_1$ and that the following inequality holds

$$|B_1(u, w)|_{V_1'} \le c\|u\|_{V_1} \|w\|_{V_1}, \qquad u, w \in V_1. \tag{3.3}$$

Moreover, the mapping $B_1 : V_1 \times V_1 \to V_1'$ is bilinear and continuous. Furthermore, since $\mathrm{div} u = 0$ for $u \in V_1$,

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(u_i w) = \left(\sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} \right) w + \sum_{i=1}^{n} u_i \frac{\partial w}{\partial x_i} = (\mathrm{div} u) w + u \cdot \nabla w = u \cdot \nabla w.$$

Hence by the integration by parts formula, see [20],

$$\begin{aligned} b_1(u, w, v) &= \int_D (u \cdot \nabla w) v \, dx = \sum_{i=1}^{n} \int_D \frac{\partial}{\partial x_i}(u_i w) v \, dx = -\sum_{i=1}^{n} \int_D u_i w \frac{\partial v}{\partial x_i} \, dx \\ &= -\int_D \left(\sum_{i=1}^{n} u_i \frac{\partial v}{\partial x_i} \right) w \, dx = -\int_D (u \cdot \nabla v) w \, dx = -b_1(u, v, w). \end{aligned}$$

In the integration by parts formula, the integral over the boundary $\sum_{i=1}^{n} \int_{\partial D} w v u_i n_i \, d\sigma = 0$, where n stands for the unit outward normal on ∂D. This follows from the periodicity conditions and the homogeneous condition on $D \cap \{x_d = 0\}$ and $D \cap \{x_d = 1\}$. Thus

$$b_1(u, w, v) = -b_1(u, v, w), \qquad u, w, v \in V_1. \tag{3.4}$$

In particular, see [22], Section 2,

$$b_1(u, v, v) = 0 \qquad u, v \in V_1. \tag{3.5}$$

Moreover, there exists a constant $c > 0$ such that if $u, w \in V_1$ and $v \in D(A_1^{\frac{\alpha}{2}})$ with $\frac{\alpha}{2} > \frac{d}{2} + 1$, then

$$\begin{aligned}|b_1(u,w,v)| = |b_1(u,v,w)| &= \left|\sum_{i=1}^{n} \int_D u_i w \frac{\partial v}{\partial x_i}\, dx\right| \\ &\le \|u\|_{L^2}\|w\|_{L^2}\|\nabla v\|_{L^\infty} \le c\|u\|_{L^2}\|w\|_{L^2}\|v\|_{D(A_1^{\frac{\alpha}{2}})}.\end{aligned}$$

Thus b_1 can be extented to the three-linear form (denoted by the same letter)

$$b_1 : H_1 \times H_1 \times D(A_1^{\frac{\alpha}{2}}) \to \mathbb{R}$$

and $|b_1(u,w,v)| \le c\|u\|_{L^2}\|w\|_{L^2}\|v\|_{D(A_1^{\frac{\alpha}{2}})}$ for $u, w \in H_1$ and $v \in D(A_1^{\frac{\alpha}{2}})$. At the same time operator B_1 can be extended to

$$B_1 : H_1 \times H_1 \to D(A_1^{-\frac{\alpha}{2}})$$

and satisfies the estimate

$$|B_1(u,w)|_{D(A_1^{-\frac{\alpha}{2}})} \le c|u|_{H_1}|w|_{H_1}, \qquad u, w \in H_1. \tag{3.6}$$

Next, let us consider the following three-linear form

$$b_2(u,\vartheta,\theta) = \int_\Omega \big(u \cdot \nabla\vartheta\big)\theta\, dx, \qquad u \in V_1. \tag{3.7}$$

As before, by the Hőlder inequality and the Sobolev embedding Theorem we have

$$\begin{aligned}|b_2(u,\vartheta,\theta)| &\le |u|_{L^4}\|\vartheta\|_{V_2}|\theta|_{L^4} \\ &\le c\|u\|_{V_1}\|\vartheta\|_{V_2}\|\theta\|_{V_2}, \quad \vartheta,\theta \in V_2\end{aligned} \tag{3.8}$$

for some constant $c > 0$. Thus the form b_2 is continuous on $V_1 \times V_2 \times V_2$. Moreover, if we define a bilinear map B_2 by $B_2(u,\vartheta) := b_2(u,\vartheta,\cdot)$, then by inequality (3.8) we infer that $B_2(u,\vartheta) \in V_2'$ for all $u \in V_1, \vartheta \in V_2$ and that the following estimate holds

$$|B_2(u,\vartheta)|_{V_2'} \le c\|u\|_{V_1}\|\vartheta\|_{V_2}, \qquad u \in V_1, \quad \vartheta \in V_2. \tag{3.9}$$

Moreover, the mapping $B_2 : V_1 \times V_2 \to V_2'$ is bilinear continuous and

$$b_2(u,\vartheta,\theta) = -b_2(u,\theta,\vartheta), \qquad u \in V_1, \quad \vartheta,\theta \in V_2. \tag{3.10}$$

In particular,

$$b_2(u,\vartheta,\vartheta)=0 \qquad u\in V_1, \quad \vartheta\in V_2. \tag{3.11}$$

Analogously, if $\frac{\alpha}{2}>\frac{d}{2}+1$, operator B_2 can be extended to a bounded bilinear map

$$B_2: H_1\times H_2 \to D(A_2^{-\frac{\alpha}{2}}).$$

In particular, the following inequality holds

$$|B_2(u,\vartheta)|_{D(A_2^{-\frac{\alpha}{2}})} \le c|u|_{H_1}|\vartheta|_{H_2}, \qquad u\in H_1,\ \vartheta\in H_2 \tag{3.12}$$

for some constant $c>0$.

Using the above notation, the Boussinesq problem can be written in the following form

$$du+\big[\nu A_1 u+B_1(u,u)-\vartheta e_d\big]\,dt = f_1(t)\,dt+G_1(u,\vartheta)\,dW_1(t), \tag{3.13}$$
$$d\vartheta+\big[\kappa A_2\vartheta+B_2(u,\vartheta)-u_d\big]\,dt = f_2(t)\,dt+G_2(u,\vartheta)\,dW_2(t). \tag{3.14}$$

with the initial conditions

$$u(0)=u_0, \qquad \vartheta(0)=\vartheta_0. \tag{3.15}$$

Thus for $\phi:=(u,\vartheta)$, we have the following equation

$$d\phi+\big[A\phi+B(\phi)+R\phi\big]\,dt = f(t)\,dt+G(\phi)\,dW(t), \tag{3.16}$$

with the initial condition

$$\phi(0)=\phi_0, \tag{3.17}$$

where $W(t)=(W_1(t),W_2(t))$, $f=\big(f_1(t),f_2(t)\big)$, $\phi_0=(u_0,\vartheta_0)$ and

$$\begin{aligned} A\phi &= (\nu A_1 u,\kappa A_2\vartheta),\\ B(\phi) &= \big(B_1(u,u),B_2(u,\vartheta)\big),\\ R\phi &= (-\vartheta e_d,-u_d),\\ G(\phi) &= \big(G_1(\phi),G_2(\phi)\big). \end{aligned}$$

Let us introduce also the following bilinear map

$$B(\phi,\psi):=\big(B_1(u,v),B_2(u,\theta)\big),$$

where $\phi=(u,\vartheta)$ and $\psi=(v,\theta)$, with the notation $B(\phi):=B(\phi,\phi)$.

Lemma 3.1.

(1) *There exists a constant* $c_1 > 0$ *such that*

$$|B(\phi,\psi)|_{V'} \le c_1 \|\phi\|_V \|\psi\|_V, \qquad \phi,\psi \in V. \tag{3.18}$$

(2) *If* $\frac{\alpha}{2} > \frac{d}{2}+1$, *then* B *can be extended to the bilinear map from* $H\times H$ *to* $D(A^{-\frac{\alpha}{2}})$. *Moreover, there exists a constant* $c_2>0$ *such that*

$$|B(\phi,\psi)|_{D(A^{-\frac{\alpha}{2}})} \le c_2 |\phi|_H |\psi|_H, \qquad \phi,\psi\in H. \tag{3.19}$$

(3) *The map* B *is locally Lipschitz continuous, i.e. for every* $r>0$ *there exists a constant* L_r *such that*

$$\big|B(\phi)-B(\tilde\phi)\big|_{V'} \le L_r \|\phi-\tilde\phi\|_V, \qquad \phi,\tilde\phi\in V, \quad \|\phi\|_V, \|\tilde\phi\|_V \le r. \tag{3.20}$$

Proof. **Ad. (1)** Let $\phi=(u,\vartheta)\in V$ and $\psi=(v,\theta)\in V$. By inequalities (3.3) and (3.9) we obtain the following estimates

$$\begin{aligned}|B(\phi,\psi)|^2_{V'} &= \big|\big(B_1(u,v),B_2(u,\theta)\big)\big|^2_{V'} = \big|B_1(u,v)\big|^2_{V_1'} + \big|B_2(u,\theta)\big|^2_{V_2'}\\ &\le c^2\|u\|^2_{V_1}\|v\|^2_{V_1} + c^2\|u\|^2_{V_1}\|\theta\|^2_{V_2} = c^2\|u\|^2_{V_1}\big(\|v\|^2_{V_1}+\|\theta\|^2_{V_2}\big)\\ &\le c_1\|\phi\|^2_V\|\psi\|^2_V\end{aligned}$$

for some constant $c_1>0$. This completes the proof of inequality (3.18).

Ad. (2) Let $\phi=(u,\vartheta)\in H$ and $\psi=(v,\theta)\in H$. Then by inequalities (3.6) and (3.12) we have the following estimates

$$\begin{aligned}|B(\phi,\psi)|^2_{D(A^{-\frac{\alpha}{2}})} &= |B_1(u,v)|^2_{D(A_1^{-\frac{\alpha}{2}})} + |B_2(u,\theta)|^2_{D(A_2^{-\frac{\alpha}{2}})}\\ &\le c^2|u|^2_{H_1}|v|^2_{H_1} + c^2|u|^2_{H_1}|\theta|^2_{H_2} \le c_2|\phi|^2_H|\psi|^2_H\end{aligned}$$

for some constant $c_2>0$. The proof of inequality (3.19) is thus complete.

Ad. (3) Let us fix $r>0$ and let $\phi=(u,\vartheta),\tilde\phi=(\tilde u,\tilde\vartheta)\in V$ be such that $\|\phi\|_V,\|\tilde\phi\|_V\le r$. We have

$$\begin{aligned}\big|B(\phi)-B(\tilde\phi)\big|^2_{V'} &= \big|\big(B_1(u,u),B_2(u,\vartheta)\big)-\big(B_1(\tilde u,\tilde u),B_2(\tilde u,\tilde\vartheta)\big)\big|^2_{V'}\\ &= \big|B_1(u,u)-B_1(\tilde u,\tilde u)\big|^2_{V_1'} + \big|B_2(u,\vartheta)-B_2(\tilde u,\tilde\vartheta)\big|^2_{V_2'}.\end{aligned}$$

We will estimate each term of the right-hand side of the above equality. By inequality (3.3) we have the following estimates

$$\begin{aligned}\left|B_1(u,u)-B_1(\tilde{u},\tilde{u})\right|_{V_1'} &\le \left|B_1(u,u-\tilde{u})\right|_{V_1'}+\left|B_1(u-\tilde{u},\tilde{u})\right|_{V_1'}\\ &\le c\|u\|_{V_1}\|u-\tilde{u}\|_{V_1}+c\|u-\tilde{u}\|_{V_1}\|\tilde{u}\|_{V_1}\\ &\le 2rc\cdot\|u-\tilde{u}\|_{V_1}\le 2rc\cdot\|\phi-\tilde{\phi}\|_V.\end{aligned}$$

By inequality (3.9) we obtain the following estimates

$$\begin{aligned}\left|B_2(u,\vartheta)-B_2(\tilde{u},\tilde{\vartheta})\right|_{V_2'} &\le \left|B_2(u,\vartheta-\tilde{\vartheta})\right|_{V_2'}+\left|B_2(u-\tilde{u},\tilde{\vartheta})\right|_{V_2'}\\ &\le c\|u\|_{V_1}\|\vartheta-\tilde{\vartheta}\|_{V_2}+c\|u-\tilde{u}\|_{V_1}\|\tilde{\vartheta}\|_{V_2}\\ &\le 2rc\cdot\|\phi-\tilde{\phi}\|_V.\end{aligned}$$

Hence

$$\left|B(\phi)-B(\tilde{\phi})\right|_{V'}^2\le 8r^2c^2\|\phi-\tilde{\phi}\|_V^2.$$

Thus the Lipschitz condition holds with $L_r=2\sqrt{2}rc$. The proof of Lemma is thus complete. □

Lemma 3.2. *Operator R has the following properties:*

(1) *For every $\phi\in H$, $R\phi\in V'$ and there exists a constant $c>0$ such that*

$$|R\phi|_{V'}\le c|\phi|_H. \tag{3.21}$$

(2) *For every $\phi\in V$:*

$$\langle R\phi|\phi\rangle\ge -|\phi|_H^2. \tag{3.22}$$

Proof. To prove the first part of the statement let $\phi=(u,\vartheta)\in H$ and $\psi=(v,\theta)\in V$. Since

$$|R\phi|^2=|(-\vartheta e_d,-u_d)|^2=\vartheta^2+u_d^2\le|\phi|^2,$$

we have the following estimates

$$\begin{aligned}\Big|\int_D(R\phi)\cdot\psi\,dx\Big| &\le \int_D|R\phi|\,|\psi|\,dx\le\int_D|\phi|\,|\psi|\,dx\\ &\le\left(\int_D|\phi|^2\,dx\right)^{\frac12}\left(\int_D|\psi|^2\,dx\right)^{\frac12}=|\phi|_H|\psi|_H\le c|\phi|_H\|\psi\|_V\end{aligned}$$

for some constant $c>0$. Thus $R\phi\in V'$ and inequality (3.21) holds.

Let us move to the second part of the statement. Let $\phi = (u, \vartheta) \in V$. Since

$$(R\phi) \cdot \phi = (-\vartheta e_d, -u_d) \cdot (u, \vartheta) = -2u_d\vartheta$$

and $2u_d\vartheta \le |\phi|^2$, thus

$$\langle R\phi|\phi\rangle = \int_D (R\phi) \cdot \phi \, dx \ge -\int_D |\phi|^2 \, dx = -|\phi|_H^2.$$

This completes the proof of inequality (3.22) and the proof of Lemma. □

Assumptions. We assume that

(A.1) $W(t)$ is a cylindrical Wiener process in a separable Hilbert space Y defined on the stochastic basis $\big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P}\big)$,

(A.2) $\phi_0 \in H$, $f \in L^2(0,T;V')$,

(A.3) The mapping $G : V \to \mathcal{L}_{HS}(Y, H)$ is Lipschitz continuous and

$$2\langle A\phi|\phi\rangle - \|G(\phi)\|^2_{\mathcal{L}_{HS}(Y,H)} \ge \eta\|\phi\|_V^2 - \lambda_0|\phi|_H^2 - \rho, \qquad \phi \in V \qquad \text{(G)}$$

for some constants λ_0, ρ and $\eta \in (\frac{4}{3}, 2]$.

Moreover, G extends to a Lipschitz continuous mapping $G : H \to \mathcal{L}_{HS}(Y, V_{-\gamma})$ for some $\gamma \ge 1$ and

$$\|G(\phi)\|^2_{\mathcal{L}_{HS}(Y,V_{-\gamma})} \le C(1 + |\phi|_H^2), \qquad \phi \in H. \qquad \text{(G}*\text{)}$$

for some $C > 0$.

Remark. The assumption that $G : V \to \mathcal{L}_{HS}(Y, H)$ is Lipschitz continuous we use only in the construction of solutions of the Galerkin equations. Since in the finite dimensional space every continuous map can be approximated by a sequence of Lipschitz continuous maps, it is sufficient to assume that G is continuous.

Furthermore, the Lipschitz continuity of the mapping $G : H \to \mathcal{L}_{HS}(Y, V_{-\gamma})$ can be replaced by the following one

$$\text{for every } \psi \in \mathcal{V} \text{ the map } H \ni u \to \langle G(u)|\psi\rangle \in Y \text{ is continuous}$$

with a slight modification of the proof of Lemma 4.6.

Definition 4. We say that there exists **a martingale solution** of the problem (3.16)-(3.17) iff there exist

- a stochastic basis $\big(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t\geq 0}, \hat{\mathbb{P}}\big)$,
- a cylindrical Wiener process $\hat{W}$ on the space Y,
- and a progressively measurable process $\phi : [0,T] \times \hat{\Omega} \to H$ with $\hat{\mathbb{P}}$ - a.e. paths
$$\phi(\cdot,\omega) \in \mathcal{C}\big([0,T], H_w\big) \cap L^2(0,T;V)$$

such that for all $t \in [0,T]$ and all $\psi \in D(A)$:

$$\begin{aligned}\big(\phi(t)|\psi\big) + \int_0^t \big\langle \phi(s)|A\psi\big\rangle\, ds + \int_0^t \big\langle B(\phi(s))|\psi\big\rangle\, ds + \int_0^t \big\langle R\phi(s)|\psi\big\rangle\, ds \\ = \big(\phi_0|\psi\big) + \int_0^t \big\langle f(s)|\psi\big\rangle\, ds + \Big\langle \int_0^t G(\phi(s))\, d\hat{W}(s)\big|\psi\Big\rangle\end{aligned} \tag{3.23}$$

the identity holds $\hat{\mathbb{P}}$ - a.s.

Here $\mathcal{C}([0,T];H_w)$ denotes the space of H - valued weakly continuous functions on $[0,T]$.

4 Existence of solutions

Theorem 4.1. *There exists a martingale solution of the problem (3.16)-(3.17) provided assumptions* (A.1)-(A.3), *are satisfied.*

4.1 Faedo-Galerkin approximation

Let $\{\psi_n\}_{n\in\mathbb{N}}$ be the orthonormal basis in H composed of eigenvectors of A. Then $\{\psi_n\}_{n\in\mathbb{N}}$ form also an orthogonal system in V. Let $H_n := \operatorname{span}\{\psi_1,...,\psi_n\}$ with the norm inherited from H and $V_n := \operatorname{span}\{\psi_1,...,\psi_n\}$ with the norm inherited from V. Let us fix $\alpha \geq \max\{\gamma, d+2\}$ and let P_n be the linear operator from $V_{-\alpha}$ to V_α defined by

$$P_n v^* := \sum_{i=1}^n \big\langle v^*|\psi_i\big\rangle \psi_i, \qquad v^* \in V_{-\alpha}.$$

Then the restriction of P_n to H is the $(\cdot|\cdot)$ orthogonal projection onto H_n and the restriction $P_{n|V} : V \to V_n$ is the $\big((\cdot|\cdot)\big)$ orthogonal projection. These restrictions will be also denoted by P_n. Moreover, in the subspace $\operatorname{span}\{\psi_1,...,\psi_n\}$ all norms are equivalent. Consider the following mapping

$$B_n(\phi) := P_n B(\chi_n(\phi), \phi), \qquad \phi \in H_n,$$

where $\chi_n : \text{span}\{\psi_1, ..., \psi_n\} \to \text{span}\{\psi_1, ..., \psi_n\}$ is defined by $\chi_n(\phi) = \theta_n(|\phi|_{V_{-\alpha}})\phi$ with $\theta_n : \mathbb{R} \to [0, 1]$ is of class $\mathcal{C}^\infty$ such that

$$\theta_n(r) = \begin{cases} 1 & \text{if} \quad r \leq n \\ 0 & \text{if} \quad r \geq n + 1. \end{cases}$$

Since $H_n \subset H$, B_n is well defined. Let us notice that

$$B_n(\phi) = \begin{cases} P_n B(\phi) & \text{if} \quad |\phi|_{V_{-\alpha}} \leq n \\ 0 & \text{if} \quad |\phi|_{V_{-\alpha}} \geq n + 1. \end{cases}$$

Moreover, by Lemma 3.1 $B_n : H_n \to H_n$ is globally Lipschitz continuous.

Consider next the classical Faedo-Galerkin approximation in the space $P_n H = H_n$

$$\begin{cases} d\phi_n(t) = -\big[P_n A\phi_n(t) + B_n\big(\phi_n\big) + P_n R(\phi_n(t)) - P_n f(t)\big]\, dt \\ \qquad\qquad + P_n G\big(\phi_n\big)\, dW(t), \quad t \in [0, T], \\ \phi_n(0) = P_n \phi_0. \end{cases} \tag{4.1}$$

Lemma 4.2. *For each $n \in \mathbb{N}$, there exists a solution of the Galerkin equation (4.1). Moreover, $\phi_n \in \mathcal{C}([0, T]; H_n)$, $\mathbb{P}$-a.s. and $\mathbb{E}[\int_0^T |\phi_n(s)|_H^q\, ds] < \infty$ for any $q \in [2, \infty)$.*

The proof is standard and thus omitted.

Using the Itô formula and the Burkholder-Davis-Gundy inequality, see [7] Lemma 7.2 or [18], we will prove the following lemma about *a priori* estimates for the solutions ϕ_n of (4.1). Let us assume

$$\begin{cases} p \in \big[2, 2 + \frac{\eta}{2-\eta}\big) & \text{if} \quad \eta \in (0, 2) \\ p \in [2, \infty) & \text{if} \quad \eta = 2. \end{cases} \tag{4.2}$$

Let us notice that in (A.3) we assume that $\eta \in (\frac{4}{3}, 2]$. However, the following lemma holds for $\eta \in (0, 2]$. The restriction $\eta > \frac{4}{3}$ is related to the nonlinear term B and it is crucial in the forthcomming passing to the limit as $n \to \infty$ considered in the next section.

Lemma 4.3. *The processes $(\phi_n)_{n \in \mathbb{N}}$ satisfy the following estimates.*

(i) *For every p satisfying (4.2) there exist positive constants* $C_1(p)$ *and* $C_2(p)$ *such that*

$$\sup_{n\in\mathbb{N}} \mathbb{E}\big(\sup_{0\le s\le T} |\phi_n(s)|_H^p \big) \le C_1(p). \tag{4.3}$$

and

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\int_0^T |\phi_n(s)|_H^{p-2} \|\phi_n(s)\|_V^2 \, ds \Big] \le C_2(p). \tag{4.4}$$

(ii) *In particular, with* $C_2 := C_2(2)$

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\int_0^T \|\phi_n(s)\|_V^2 \, ds \Big] \le C_2. \tag{4.5}$$

Proof. See Appendix A. □

4.2 Tightness

Let $\mathbb{H} := H$ and $\mathbb{V} := V$ and let $U := D(A^{\frac{\alpha}{2}})$ with $\alpha \ge \max\{\gamma, d+2\}$. For each $n \in \mathbb{N}$, the solution ϕ_n of the Galerkin equation defines a measure $\mathcal{L}(\phi_n)$ on $(\mathcal{Z}, \mathcal{T})$, where $\mathcal{Z}$ is the space defined by (2.6) with $q = 2$. Using the tigthness criterion contained in Corollary 2.6 we will prove that the set of measures $\{\mathcal{L}(\phi_n), n \in \mathbb{N}\}$ is tight. The main difficulty lies in checking the Aldous condition which corresponds to the modulus of continuity. To this end we need suitable estimates.

The idea of our approach is similar to that used by Flandoli and Gątarek [10], where the tightness is proven by means of some compactness results in fractional Sobolev spaces. In this way, instead of the estimates on the modulus of continuity there are estimates in suitable fractional Sobolev spaces. Because the tightness criterion we used is different from the one than used in [10], we are forced to find different a'priori estimates on the sequence of approximating solutions.

Lemma 4.4. *The set of measures* $\{\mathcal{L}(\phi_n), n \in \mathbb{N}\}$ *is relatively weakly compact on* $(\mathcal{Z}, \mathcal{T})$.

Proof. We use Corollary 2.6 with $\mathbb{H} := H$ and $\mathbb{V} := V$. By estimates (4.3) and (4.5) conditions (a), (b) are satisfied. Thus it is sufficient to prove that the sequence $(\phi_n)_{n\in\mathbb{N}}$ satisfies the Aldous condition **[A]** in $U' = D(A^{-\frac{\alpha}{2}}) = V_{-\alpha}$.

Let $(\tau_n)_{n\in\mathbb{N}}$ be a sequence of stopping times such that $0 \le \tau_n \le T$. Let us write (4.1) in the following form

$$\begin{aligned}\phi_n(t) &= P_n\phi_0 - \int_0^t A\phi_n(s)\,ds - \int_0^t P_nB\big(\phi_n(s),\phi_n(s)\big)\,ds - \int_0^t P_nR\phi_n(s)\,ds \\ &\quad + \int_0^t P_nf(s)\,ds + \int_0^t P_nG(\phi_n(s))\,dW(s) \\ &=: J_n^1 + J_n^2(t) + J_n^3(t) + J_n^4(t) + J_n^5(t) + J_n^6(t), \qquad t\in[0,T]. \end{aligned} \tag{4.6}$$

Let $\theta > 0$. First, we make some estimates for each term of the above equality.

Ad. J_n^2**.** Since $A : V \to V'$ and $|A(\phi)|_{V'} \le \|\phi\|_V$ and $V' \hookrightarrow U'$, then by the Hőlder inequality and (4.5) we have the following estimates

$$\begin{aligned}\mathbb{E}\big[\big|J_n^2(\tau_n+\theta) - J_n^2(\tau_n)\big|_{U'}\big] &= \mathbb{E}\bigg[\bigg|\int_{\tau_n}^{\tau_n+\theta} A\phi_n(s)\,ds\bigg|_{U'}\bigg] \\ &\le c\,\mathbb{E}\bigg[\bigg|\int_{\tau_n}^{\tau_n+\theta} A\phi_n(s)\,ds\bigg|_{V'}\bigg] \le c\,\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta} \big|A\phi_n(s)\big|_{V'}\,ds\bigg] \\ &\le c\,\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta} \|\phi_n(s)\|_V\,ds\bigg] \le c\,\mathbb{E}\bigg[\theta\int_0^T \|\phi_n(s)\|_V^2\,ds\bigg] \\ &\le c\,C_2\cdot\theta =: c_2\cdot\theta. \end{aligned} \tag{4.7}$$

Ad. J_n^3**.** Similarly, since $B : H\times H \to U'$ is bilinear and continuous, then by (4.3) we have the following estimates

$$\begin{aligned}\mathbb{E}\big[\big|J_n^3(\tau_n+\theta) - J_n^3(\tau_n)\big|_{U'}\big] &= \mathbb{E}\bigg[\bigg|\int_{\tau_n}^{\tau_n+\theta} B_n\big(\phi_n(s)\big)\,ds\bigg|_{U'}\bigg] \\ &\le \mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta} \big|B\big(\phi_n(s)\big)\big|_{U'}\,ds\bigg] \le \mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta} \|B\|\cdot\big|\phi_n(s)\big|_H^2\,ds\bigg] \\ &\le \|B\|\cdot\mathbb{E}\big[\sup_{s\in[0,T]}\big|\phi_n(s)\big|_H^2\big]\cdot\theta \le \|B\|\,C_1(2)\cdot\theta =: c_3\cdot\theta, \end{aligned} \tag{4.8}$$

where $\|B\|$ stands for the norm of $B : H\times H \to U'$.

Ad. J_n^4**.** By Lemma 3.2 and (4.3) we have the following estimates

$$\begin{aligned}\mathbb{E}\big[\big|J_n^4(\tau_n+\theta) - J_n^4(\tau_n)\big|_{U'}\big] &= \mathbb{E}\bigg[\bigg|\int_{\tau_n}^{\tau_n+\theta} P_nR\phi_n(s)\,ds\bigg|_{U'}\bigg] \\ &\le c\,\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta} \big|R\phi_n(s)\big|_{V'}\,ds\bigg] \le c\,\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta} \big|\phi_n(s)\big|_H\,ds\bigg] \\ &\le \theta^{\frac12}\,\mathbb{E}\bigg[\bigg(\int_{\tau_n}^{\tau_n+\theta} \big|\phi_n(s)\big|_H^2\,ds\bigg)^{\frac12}\bigg] \le \theta\,\mathbb{E}\big[\sup_{s\in[0,T]}\big|\phi_n(s)\big|_H^2\big] \\ &\le C_1(2)\theta =: c_4\cdot\theta. \end{aligned} \tag{4.9}$$

Ad. J_n^5**.** Since $f \in L^2(0,T;V')$,

$$\begin{aligned}\mathbb{E}\big[\big|J_n^5(\tau_n+\theta)-J_n^5(\tau_n)\big|_{U'}\big] &= \mathbb{E}\bigg[\bigg|\int_{\tau_n}^{\tau_n+\theta} P_n f(s)\,ds\bigg|_{U'}\bigg]\\ &\le c\,\mathbb{E}\bigg[\bigg|\int_{\tau_n}^{\tau_n+\theta} P_n f(s)\,ds\bigg|_{V'}\bigg] \le c\;\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta} \big|f(s)\big|_{V'}\,ds\bigg]\\ &\le c\cdot\theta\cdot\mathbb{E}\bigg[\int_0^T \big|f(s)\big|_{V'}^2\,ds\bigg] = c\cdot\theta\cdot\|f\|_{L^2(0,T;V')}^2 =: c_5\cdot\theta.\end{aligned} \tag{4.10}$$

Ad. J_n^6**.** Since $U' = D(A^{-\frac{\alpha}{2}}) = V_{-\alpha}$ and $\alpha > \gamma$, then $V_{-\gamma} \hookrightarrow U'$. By the continuity of the embedding $\mathcal{L}_{HS}(Y,V_{-\gamma}) \hookrightarrow \mathcal{L}(Y,V_{-\gamma})$, (G$*$) and (4.3) we have the following estimates

$$\begin{aligned}&\mathbb{E}\big[\big|J_n^6(\tau_n+\theta)-J_n^6(\tau_n)\big|_{U'}^2\big]\\ &\quad= \mathbb{E}\bigg[\bigg|\int_0^{\tau_n+\theta} P_n G(\phi_n(s))\,dW(s) - \int_0^{\tau_n} P_n G(\phi_n(s))\,dW(s)\bigg|_{U'}^2\bigg]\\ &\quad= \mathbb{E}\bigg[\bigg|\int_{\tau_n}^{\tau_n+\theta} P_n G(\phi_n(s))\,dW(s)\bigg|_{U'}^2\bigg]\\ &\quad\le c\;\mathbb{E}\bigg[\bigg|\int_{\tau_n}^{\tau_n+\theta} P_n G(\phi_n(s))\,dW(s)\bigg|_{V_{-\gamma}}^2\bigg]\\ &\quad\le c\;\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta} \|G(\phi_n(s))\|_{\mathcal{L}_{HS}(Y,V_{-\gamma})}^2\,ds\bigg]\\ &\quad\le const\cdot\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+\theta} (1+|\phi_n(s)|_H^2)\,ds\bigg]\\ &\quad\le const\cdot\theta\cdot\mathbb{E}\big[\sup_{s\in[0,T]}\big|\phi_n(s)\big|_H^2\big] \le const\cdot C_1(2)\cdot\theta =: c_6\cdot\theta.\end{aligned} \tag{4.11}$$

Let us fix $\eta > 0$ and $\varepsilon > 0$. By the Chebyshev inequality and estimates (4.7)-(4.10) we obtain

$$\begin{aligned}\mathbb{P}\big(\big\{\big|J_n^i(\tau_n+\theta)-J_n^i(\tau_n)\big|_{U'} \ge \eta\big\}\big) &\le \frac{1}{\eta}\,\mathbb{E}\big[\big|J_n^i(\tau_n+\theta)-J_n^i(\tau_n)\big|_{U'}\big]\\ &\le \frac{c_i\cdot\theta}{\eta}, \qquad n\in\mathbb{N},\end{aligned}$$

where $i = 1,2,3,4,5$. Let $\delta_i := \frac{\eta}{c_i}\cdot\varepsilon$. Then

$$\sup_{n\in\mathbb{N}}\sup_{1\le\theta\le\delta_i}\mathbb{P}\big\{\big|J_n^i(\tau_n+\theta)-J_n^i(\tau_n)\big|_{U'} \ge \eta\big\} \le \varepsilon, \quad i=1,2,3,4,5.$$

For the noise term, by (4.11) we have

$$\mathbb{P}\big(\big\{\big|J_n^6(\tau_n+\theta)-J_n^6(\tau_n)\big|_{U'}\ge\eta\big\}\big)\le\frac{1}{\eta^2}\,\mathbb{E}\big[\big|J_n^6(\tau_n+\theta)-J_n^6(\tau_n)\big|_{U'}^2\big]$$
$$\le\frac{c_6\cdot\theta}{\eta^2},\qquad n\in\mathbb{N}.$$

Let $\delta_6:=\frac{\eta^2}{c_6}\cdot\varepsilon$. Then

$$\sup_{n\in\mathbb{N}}\sup_{1\le\theta\le\delta_6}\mathbb{P}\big\{\big|J_n^6(\tau_n+\theta)-J_n^6(\tau_n)\big|_{U'}\ge\eta\big\}\le\varepsilon.$$

Since condition **[A]** holds for each term J_n^i, $i=1,2,3,4,5,6$, we infer that it holds also for (ϕ_n). This completes the proof of lemma. □

4.3 Proof of Theorem 4.1

We will construct a martingale solution using the method used by Da Prato and Zabczyk in [7], Section 8.

By Lemma 4.4 the set of measures $\big\{\mathcal{L}(\phi_n), n\in\mathbb{N}\big\}$ is relatively weakly compact on $(\mathcal{Z},\mathcal{T})$. Thus we can find a subsequence, still denoted by (ϕ_n) such that

$$\mathcal{L}(\phi_n)\text{ converges weakly on }\mathcal{Z}\quad\text{ as }n\to\infty.$$

Thus

$$\mathcal{L}(\phi_n)\text{ converges weakly on }\mathcal{C}([0,T];U')\cap L^2(0,T;H).$$

as $n\to\infty$.

By the Skorokhod Theorem, see [7], there exist

- a stochastic basis $\big(\tilde{\Omega},\tilde{\mathcal{F}},\{\tilde{\mathcal{F}}_t\}_{t\in[0,T]},\tilde{\mathbb{P}}\big)$ and, on this basis,
- $\mathcal{C}([0,T];U')\cap L^2(0,T;H)$ - valued random variables $\tilde{\phi}$, $\tilde{\phi}_n$, $n\ge 1$

such that

$$\tilde{\phi}_n\text{ has the same law as }\phi_n\text{ on }\mathcal{C}([0,T];U')\cap L^2(0,T;H),$$
$$\text{and }\tilde{\phi}_n\to\tilde{\phi}\text{ in }\mathcal{C}([0,T];U')\cap L^2(0,T;H),\quad\tilde{\mathbb{P}}\text{ - a.s.}$$

Since, by Lemma 4.2, $\phi_n\in\mathcal{C}([0,T];P_nH)$, $\tilde{\phi}_n$ and ϕ_n have the same laws, and $\mathcal{C}([0,T];P_nH)$ is a Borel subset of $\mathcal{C}([0,T];U')\cap L^2(0,T;H)$, then we have

$$\mathcal{L}(\tilde{\phi}_n)\big(\mathcal{C}([0,T];P_nH)\big)=1,\qquad n\ge 1.$$

Since $\tilde{\phi}_n$ and ϕ_n have the same laws, and $\mathcal{C}([0,T];P_nH)$ is a Borel subset of $\mathcal{C}([0,T];U')\cap L^2(0,T;H)$, thus by (4.3) and (4.5) we have the following estimates

$$\mathbb{E}\Big(\sup_{0\le s\le T}\big|\tilde{\phi}_n(s)\big|_H^p\Big)\le C_1(p) \tag{4.12}$$

$$\mathbb{E}\Big[\int_0^T \big\|\tilde{\phi}_n(s)\big\|_V^2\,ds\Big]\le C_2. \tag{4.13}$$

for all $n\in\mathbb{N}$ and all p satisfying condition (4.2).

For each $n\ge 1$, let us consider a process $\tilde{M}_n$ with trajectories in $\mathcal{C}([0,T];H)$ defined by

$$\begin{aligned}\tilde{M}_n(t) =&\tilde{\phi}_n(t)\,-P_n\tilde{\phi}(0)+\int_0^t A\tilde{\phi}_n(s)\,ds+\int_0^t B_n\big(\tilde{\phi}_n(s)\big)\,ds\\ &+\int_0^t P_nR\big(\tilde{\phi}_n(s)\big)\,ds-\int_0^t P_nf(s)\,ds,\quad t\in[0,T].\end{aligned}$$

$\tilde{M}_n$ is a square integrable martingale with respect to the filtration $\tilde{\mathbb{F}}_n=(\tilde{\mathcal{F}}_{n,t})$, where $\tilde{\mathcal{F}}_{n,t}=\sigma\{\tilde{\phi}_n(s),\ s\le t\}$, with the quadratic variation

$$\big\langle\!\big\langle\tilde{M}_n\big\rangle\!\big\rangle_t=\int_0^t P_nG(\tilde{\phi}_n(s))\big(G(\tilde{\phi}_n(s))\big)^*P_n\,ds. \tag{4.14}$$

Indeed, since $\tilde{\phi}_n$ and ϕ_n have the same laws, for all $s\le t\in[0,T]$ all functions h bounded continuous on $L^2(0,s;H)\cap\mathcal{C}([0,s];U')$ and all $\psi,\zeta\in D(A^{\frac{\alpha}{2}})$, we have

$$\mathbb{E}\big[\big\langle\tilde{M}_n(t)-\tilde{M}_n(s)|\psi\big\rangle\,h\big(\tilde{\phi}_{n|[0,s]}\big)\big]=0 \tag{4.15}$$

and

$$\begin{aligned}\mathbb{E}\Bigg[\Bigg(&\big\langle\tilde{M}_n(t)|\psi\big\rangle\big\langle\tilde{M}_n(t)|\zeta\big\rangle-\big\langle\tilde{M}_n(s)|\psi\big\rangle\big\langle\tilde{M}_n(s)|\zeta\big\rangle\\ &-\int_s^t\Big\langle G(\tilde{\phi}_n(\sigma))^*P_n\psi\big|G(\tilde{\phi}_n(\sigma))^*P_n\zeta\Big\rangle\,d\sigma\Bigg)\cdot h\big(\tilde{\phi}_{n|[0,s]}\big)\Bigg]=0.\end{aligned} \tag{4.16}$$

We will pass to the limit in (4.15) and (4.16) as $n\to\infty$. All terms in (4.15) and (4.16) are uniformly integrable in ω, and converge $\tilde{\mathbb{P}}$ - a.s. The main difficulties in (4.16) occur in terms containing the nonlinearity B and in the term corresponding to the quadratic variation of the martingale $\tilde{M}_n$. We consider these problems in the following two lemmas. In Lemma 4.5 the assumption that $\eta\in(\frac{4}{3},2]$ will be of crucial importance.

Lemma 4.5. *If $\eta \in (\frac{4}{3}, 2]$ then for any $\psi, \zeta \in D(A^{\frac{\alpha}{2}})$:*

$$\lim_{n\to\infty} \mathbb{E}\Big[\int_s^t \big\langle B_n(\tilde{\phi}_n(\sigma))|\psi\big\rangle\, d\sigma \cdot \int_s^t \big\langle B_n(\tilde{\phi}_n(\sigma))|\zeta\big\rangle\, d\sigma \cdot h\big(\tilde{\phi}_{n|[0,s]}\big)\Big]$$
$$= \mathbb{E}\Big[\int_s^t \big\langle B(\tilde{\phi}(\sigma))|\psi\big\rangle\, d\sigma \cdot \int_s^t \big\langle B(\tilde{\phi}(\sigma))|\zeta\big\rangle\, d\sigma \cdot h\big(\tilde{\phi}_{|[0,s]}\big)\Big].$$

Proof. Let us denote

$$g_n(\omega) := \int_s^t \big\langle B_n(\tilde{\phi}_n(\sigma,\omega))|\psi\big\rangle\, d\sigma \cdot \int_s^t \big\langle B_n(\tilde{\phi}_n(\sigma,\omega))|\zeta\big\rangle\, d\sigma \cdot h\big(\tilde{\phi}_{n|[0,s]}(\omega)\big)$$
$$g(\omega) := \int_s^t \big\langle B(\tilde{\phi}(\sigma,\omega))|\psi\big\rangle\, d\sigma \cdot \int_s^t \big\langle B(\tilde{\phi}(\sigma,\omega))|\zeta\big\rangle\, d\sigma \cdot h\big(\tilde{\phi}_{|[0,s]}(\omega)\big), \quad \omega \in \tilde{\Omega}.$$

We will prove that the functions $\{g_n\}_{n\in\mathbb{N}}$ are uniformly integrable and $\lim_{n\to\infty} g_n(\omega) = g(\omega)$ for $\tilde{\mathbb{P}}$-almost all $\omega \in \tilde{\Omega}$.

Uniform integrability. It is sufficient to show that

$$\sup_{n\geq 1} \mathbb{E}\big[|g_n|^r\big] < \infty \tag{4.17}$$

for some $r > 1$. In fact, we will prove that the above condition holds with every
$r \in \big(1, \frac{1}{2} + \frac{\eta}{4(2-\eta)}\big)$ if $\frac{4}{3} < \eta < 2$ (notice that because of the assumption on η this interval is nonempty), and with arbitrary $r > 1$ if $\eta = 2$.

By Lemma 3.1,(2) we have the following inequality

$$\int_s^t |\big\langle B_n(\tilde{\phi}_n(\sigma,\omega))|\psi\big\rangle|\, d\sigma \leq c_2 \int_s^t |\tilde{\phi}_n(\sigma,\omega)|_H^2\, d\sigma \cdot \|\psi\|_{V_\alpha}, \quad n \geq 1.$$

Thus

$$|g_n(\omega)|^r \leq \|h\|_{L^\infty}^r c_2^2 \|\psi\|_{V_\alpha} \|\zeta\|_{V_\alpha} \bigg(\int_s^t |\tilde{\phi}_n(\sigma,\omega)|_H^2\, d\sigma\bigg)^{2r}$$
$$\leq C \int_s^t |\tilde{\phi}_n(\sigma,\omega)|_H^{4r}\, d\sigma \leq C \sup_{\sigma\in[0,T]} |\tilde{\phi}_n(\sigma,\omega)|_H^{4r}, \qquad n \geq 1,$$

where C stands for some constant. Since $4r$ satisfies condition (4.2), then by (4.12)

$$\sup_{n\geq 1} \mathbb{E}\big[|g_n|^r\big] \leq C\, \mathbb{E}\big[\sup_{\sigma\in[0,T]} |\tilde{\phi}_n(\sigma,\omega)|_H^{4r}\big] \leq CC_1(4r),$$

which completes the proof of (4.17).

Pointwise convergence in ω. Let us fix $\omega \in \tilde{\Omega}$ such that $\tilde{\phi}_n(\cdot,\omega) \to \tilde{\phi}(\cdot,\omega)$ in $L^2(0,T,H) \cap \mathcal{C}([0,T];U')$. By the continuity of h it follows that $\lim_{n\to\infty} h\big(\tilde{\phi}_{n|[0,s]}(\omega)\big) = h\big(\tilde{\phi}_{|[0,s]}(\omega)\big)$. Since $\tilde{\phi}_n(\cdot,\omega) \to \tilde{\phi}(\cdot,\omega)$ in $\mathcal{C}([0,T];U')$, the sequence $(\tilde{\phi}_n(\cdot,\omega))_{n\in\mathbb{N}}$ is bounded in $\mathcal{C}([0,T];U')$. Suppose that

$$\big|\tilde{\phi}_n(\cdot,\omega)\big|_{\mathcal{C}([0,T];U')} \le N, \qquad n \ge 1$$

for some $N > 0$. Then $\chi_n(\tilde{\phi}_n(\cdot,\omega)) = \tilde{\phi}_n(\cdot,\omega)$ for $n > N$ and

$$B_n(\tilde{\phi}_n(\cdot,\omega)) = P_n B\big(\chi_n(\tilde{\phi}_n(\cdot,\omega)), \tilde{\phi}_n(\cdot,\omega)\big) = P_n B\big(\tilde{\phi}_n(\cdot,\omega)\big), \qquad n > N.$$

On the other hand, since $\tilde{\phi}_n(\cdot,\omega) \to \tilde{\phi}(\cdot,\omega)$ in $L^2(0,T,H)$, then by Lemma 7.1 in Appendix B we infer that $\lim_{n\to\infty} g_n(\omega) = g(\omega)$. The proof of Lemma is thus complete. □

Lemma 4.6. (Convergence in quadratic variation). For any $\psi, \zeta \in D(A^{\frac{\alpha}{2}})$:

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}\bigg[\bigg(\int_s^t \Big\langle G(\tilde{\phi}_n(\sigma))^* P_n\psi \Big| G(\tilde{\phi}_n(\sigma))^* P_n\zeta \Big\rangle \, d\sigma\bigg)\cdot h\big(\tilde{\phi}_{n|[0,s]}\big)\bigg] \\ = \mathbb{E}\bigg[\bigg(\int_s^t \Big\langle G(\tilde{\phi}(\sigma))^* \psi \Big| G(\tilde{\phi}(\sigma))^* \zeta \Big\rangle \, d\sigma\bigg)\cdot h\big(\tilde{\phi}_{|[0,s]}\big)\bigg].\end{aligned}$$

Proof. Let us put

$$f_n(\omega) := \bigg(\int_s^t \Big\langle G(\tilde{\phi}_n(\sigma,\omega))^* P_n\psi \Big| G(\tilde{\phi}_n(\sigma,\omega))^* P_n\zeta \Big\rangle \, d\sigma\bigg)\cdot h\big(\tilde{\phi}_{n|[0,s]}\big), \quad \omega \in \tilde{\Omega}. \tag{4.18}$$

We will prove that these functions are uniformly integrable and convergent $\tilde{\mathbb{P}}$-a.s.

Uniform integrability. It is sufficient to prove that

$$\sup_{n\ge 1} \mathbb{E}\big[|f_n|^r\big] < \infty$$

for some $r > 1$. We will prove that the above condition holds with every $r \in \big(1, 1 + \frac{\eta}{2(2-\eta)}\big)$ if $0 < \eta < 2$ and with arbitrary $r > 1$ if $\eta = 2$.

Since $\mathcal{L}_{HS}(Y,V') \hookrightarrow \mathcal{L}(Y,V')$, then by (G$*$) we have

$$\begin{aligned}\big|G(\tilde{\phi}_n(\sigma,\omega))^* P_n\zeta\big|_Y &\le \|G(\tilde{\phi}_n(\sigma,\omega))\|_{\mathcal{L}(Y,V')} \cdot \|P_n\zeta\|_V \\ &\le \sqrt{C\big(|\tilde{\phi}_n(\sigma,\omega)|_H^2 + 1\big)}\, \|\zeta\|_V.\end{aligned}$$

Hence we have the following estimates

$$
\begin{aligned}
|f_n|^r &= \left| \left(\int_s^t \left\langle G(\tilde{\phi}_n(\sigma,\omega))^* P_n\psi \middle| G(\tilde{\phi}_n(\sigma,\omega))^* P_n\zeta \right\rangle d\sigma \right) \cdot h\big(\tilde{\phi}_{n|[0,s]}\big) \right|^r \\
&\le \|h\|_{L^\infty}^r \left(\int_s^t \big|G(\tilde{\phi}_n(\sigma,\omega))^* P_n\psi\big|_Y \cdot \big|G(\tilde{\phi}_n(\sigma,\omega))^* P_n\zeta\big|_Y \, d\sigma \right)^r \\
&\le C^r \|h\|_{L^\infty}^r \cdot \|\psi\|_V^r \cdot \|\zeta\|_V^r \cdot \left(\int_s^t \big(|\tilde{\phi}_n(\sigma,\omega)|_H^2 + 1 \big) \, d\sigma \right)^r .
\end{aligned}
$$

Using the Hőlder inequality, we obtain the following estimates

$$
\begin{aligned}
\left(\int_s^t \big(|\tilde{\phi}_n(\sigma,\omega)|_H^2 + 1 \big) \, d\sigma \right)^r &\le (t-s)^{r-1} \cdot \int_s^t \big(|\tilde{\phi}_n(\sigma,\omega)|_H^2 + 1 \big)^r \, d\sigma \\
&\le C \cdot \sup_{\sigma\in[0,T]} \big(|\tilde{\phi}_n(\sigma,\omega)|_H^{2r} + 1 \big)
\end{aligned}
$$

for some $C > 0$. Thus

$$
|f_n|_H^r \le \tilde{C} \cdot \sup_{\sigma\in[0,T]} \big(|\tilde{\phi}_n(\sigma,\omega)|_H^{2r} + 1 \big)
$$

for some $\tilde{C}$. Hence by (4.12)

$$
\mathbb{E}\big[|f_n|_H^r\big] \le \tilde{C} \cdot \mathbb{E}\big[\sup_{\sigma\in[0,T]} |\tilde{\phi}_n(\sigma,\omega)|_H^{2r} + 1 \big] \le \tilde{C}\big(C_1(2r) + 1\big) < \infty, \qquad n \in \mathbb{N},
$$

which completes the proof of (4.18).

Pointwise convergence in ω. Let us fix ω such that $\tilde{\phi}_n(\cdot,\omega) \to \tilde{\phi}(\cdot,\omega)$ in $L^2(0,T,H) \cap \mathcal{C}([0,T];U')$. Then, in particular, the sequence $(\tilde{\phi}_n(\cdot,\omega))$ is bounded in $L^2(0,T;H)$. By the continuity of h, it follows that $\lim_{n\to\infty} h\big(\tilde{\phi}_{n|[0,s]}(\omega)\big) = h\big(\tilde{\phi}_{|[0,s]}(\omega)\big)$. We will prove that

$$
\begin{aligned}
\lim_{n\to\infty} \int_s^t & \left\langle G(\tilde{\phi}_n(\sigma,\omega))^* P_n\psi \middle| G(\tilde{\phi}_n(\sigma,\omega))^* P_n\zeta \right\rangle d\sigma \\
&= \int_s^t \left\langle G(\tilde{\phi}(\sigma,\omega))^* \psi \middle| G(\tilde{\phi}(\sigma,\omega))^* \zeta \right\rangle d\sigma.
\end{aligned}
$$

Let us remark that it is sufficient to prove that

$$
G(\tilde{\phi}_n(\cdot,\omega))^* P_n\psi \to G(\tilde{\phi}(\cdot,\omega))^* \psi \quad \text{in} \quad L^2(s,t;Y).
$$

We have the following inequalities

$$\begin{aligned}
&\int_s^t \left|G(\tilde\phi_n(\sigma,\omega))^* P_n\psi - G(\tilde\phi(\sigma,\omega))^*\psi\right|_Y^2 d\sigma \\
&\le \int_s^t \left(\left|G(\tilde\phi_n(\sigma,\omega))^*(P_n\psi-\psi)\right|_Y + \left|G(\tilde\phi_n(\sigma,\omega))^*\psi - G(\tilde\phi(\sigma,\omega))^*\psi\right|_Y \right)^2 d\sigma \\
&\le 2\left|P_n\psi-\psi\right|_{V_\gamma}^2 \cdot \int_s^t \left|G(\tilde\phi_n(\sigma,\omega))^*\right|_{\mathcal{L}(V_{-\gamma},Y)}^2 d\sigma \\
&+ 2\int_s^t \left|G(\tilde\phi_n(\sigma,\omega))^*\psi - G(\tilde\phi(\sigma,\omega))^*\psi\right|_Y^2 d\sigma =: 2\{I_1(n)+I_2(n)\}.
\end{aligned}$$

Let us consider the term $I_1(n)$ on the right hand side of the above inequality. Since

$$P_n\psi \to \psi \quad \text{for} \quad \psi \in V_\gamma,$$

then by (G$*$) the continuity of the embedding $\mathcal{L}_{HS}(Y,V_{-\gamma}) \hookrightarrow \mathcal{L}(Y,V_{-\gamma})$ and the boundedness of the sequence $(\tilde\phi_n(\cdot,\omega))$ in $L^2(0,T;H)$, we have the following estimates

$$\int_s^t \left|G(\tilde\phi_n(\sigma,\omega))^*\right|_{\mathcal{L}(V_{-\gamma},Y)}^2 d\sigma \le C\int_0^T \left(\left|\tilde\phi_n(\sigma,\omega)\right|_H^2 + 1\right) d\sigma \le \tilde C, \quad n\in\mathbb{N},$$

for some constant $\tilde C > 0$. Thus

$$\lim_{n\to\infty} I_1(n) = \lim_{n\to\infty}\int_s^t \left|G(\tilde\phi_n(\sigma,\omega))^*\right|_{\mathcal{L}(V_{-\gamma},Y)}^2 \cdot \left|P_n\psi-\psi\right|_{V_\gamma}^2 d\sigma = 0.$$

Let us move to the term $I_2(n)$. By the second part of assumption (A.3), G extends to a Lipschitz continuous mapping $G: H\to\mathcal{L}_{HS}(Y,V_{-\gamma})$. Thus

$$\begin{aligned}
\|G(\tilde\phi_n(\sigma,\omega))^*\psi - &G(\tilde\phi(\sigma,\omega))^*\psi\|_Y \\
&\le \left|G(\tilde\phi_n(\sigma,\omega)) - G(\tilde\phi(\sigma,\omega))\right|_{\mathcal{L}(Y,V_{-\gamma})} \cdot \|\psi\|_{V_\gamma} \\
&\le \left|G(\tilde\phi_n(\sigma,\omega)) - G(\tilde\phi(\sigma,\omega))\right|_{\mathcal{L}_{HS}(Y,V_{-\gamma})} \cdot \|\psi\|_{V_\gamma} \\
&\le L_G\left|\tilde\phi_n(\sigma,\omega) - \tilde\phi(\sigma,\omega)\right|_H \cdot \|\psi\|_{V_\gamma},
\end{aligned}$$

where L_G stand for the Lipschitz constant. Hence

$$\begin{aligned}
\int_s^t \left\|G(\tilde\phi_n(\sigma,\omega))^*\psi - \right.&\left. G(\tilde\phi(\sigma,\omega))^*\psi\right\|_Y^2 d\sigma \\
&\le L_G^2 \int_s^t \left|\tilde\phi_n(\sigma,\omega) - \tilde\phi(\sigma,\omega)\right|_H^2 d\sigma \cdot \|\psi\|_{V_\gamma}^2 \\
&\le L_G^2 \cdot \left\|\tilde\phi_n(\cdot,\omega) - \tilde\phi(\cdot,\omega)\right\|_{L^2(0,T;H)}^2 \cdot \|\psi\|_{V_\gamma}^2.
\end{aligned}$$

Since $\tilde{\phi}_n(\cdot,\omega) \to \tilde{\phi}(\cdot,\omega)$ in $L^2(0,T;H)$ as $n \to \infty$, we infer that

$$\lim_{n\to\infty} I_2(n) = \lim_{n\to\infty} \int_s^t \left\| G(\tilde{\phi}_n(\sigma,\omega))^* \psi - G(\tilde{\phi}(\sigma,\omega))^* \psi \right\|_Y^2 d\sigma = 0.$$

This completes the proof of lemma. □

Continuation of the proof of Theorem 4.1. By lemmas 4.5 and 4.6 we infer that for all $s \le t \in [0,T]$ all functions h bounded continuous on $L^2(0,s;H) \cap \mathcal{C}([0,s];U')$ and all $\psi, \zeta \in D(A^{\frac{\alpha}{2}})$ the following equalities hold

$$\mathbb{E}\big[\big\langle \tilde{M}(t) - \tilde{M}(s) | \psi \big\rangle \, h\big(\tilde{\phi}_{|[0,s]}\big)\big] = 0 \tag{4.19}$$

and

$$\begin{aligned}\mathbb{E}\Bigg[\bigg(&\big\langle \tilde{M}(t)|\psi\big\rangle\big\langle \tilde{M}(t)|\zeta\big\rangle - \big\langle \tilde{M}(s)|\psi\big\rangle\big\langle \tilde{M}(s)|\zeta\big\rangle \\ &- \int_s^t \Big\langle G(\tilde{\phi}(\sigma))^*\psi \Big| G(\tilde{\phi}(\sigma))^*\zeta \Big\rangle \, d\sigma\bigg) \cdot h\big(\tilde{\phi}_{|[0,s]}\big)\Bigg] = 0,\end{aligned} \tag{4.20}$$

where $\tilde{M}$ is a $D(A^{-\frac{\alpha}{2}})$-valued process defined by

$$\begin{aligned}\tilde{M}(t) =& \tilde{\phi}(t) - \tilde{\phi}(0) + \int_0^t A\tilde{\phi}(s)\,ds + \int_0^t B\big(\tilde{\phi}(s)\big)\,ds \\ &+ \int_0^t R\big(\tilde{\phi}(s)\big)\,ds - \int_0^t f(s)\,ds, \qquad t \in [0,T].\end{aligned} \tag{4.21}$$

By (4.19) and (4.20) with $\psi, \zeta \in D(A^{\frac{\alpha}{2}})$, we see that

The process $A^{-\frac{\alpha}{2}}\tilde{M}(t),\ t \in [0,T]$ is a continuous square integrable H-valued martingale with respect to the filtration $\tilde{\mathbb{F}} = \big(\tilde{\mathcal{F}}_t\big)$, where $\mathcal{F}_t = \sigma\{\tilde{\phi}(s), s \le t\}$ with quadratic variation

$$\big\langle\big\langle A^{-\frac{\alpha}{2}}\tilde{M}\big\rangle\big\rangle_t = \int_0^t A^{-\frac{\alpha}{2}} G(\tilde{\phi}(s)) G(\tilde{\phi}(s))^* A^{-\frac{\alpha}{2}}\,ds, \qquad t \in [0,T]. \tag{4.22}$$

In particular, the continuity of the process $A^{-\frac{\alpha}{2}}\tilde{M}(t)$, $t \in [0,T]$ follows from the facts that $\tilde{\phi} \in \mathcal{C}([0,T];D(A^{-\frac{\alpha}{2}}))$ and that the nonlinear term $B(\tilde{\phi}) \in L^1\big(0,T;D(A^{-\frac{\alpha}{2}})\big)$. Thus the integral in (4.22) as a function of t is continuous.

By the representation theorem for martingales, see [7], there exist

- a stochastic basis $\big(\tilde{\tilde{\Omega}}, \tilde{\tilde{\mathcal{F}}}, \{\tilde{\tilde{\mathcal{F}}}_t\}_{t\geq 0}, \tilde{\tilde{\mathbb{P}}}\big)$,
- a cylindrical Wiener process $\tilde{\tilde{W}}(t)$ defined on this basis
- and a progressively measurable proces $\tilde{\tilde{\phi}}(t)$ such that

$$\begin{aligned} &A^{-\frac{\alpha}{2}}\tilde{\tilde{\phi}}(t) \, - A^{-\frac{\alpha}{2}}\tilde{\tilde{\phi}}(0) + A^{-\frac{\alpha}{2}}\int_0^t A\tilde{\tilde{\phi}}(s)\,ds + A^{-\frac{\alpha}{2}}\int_0^t B\big(\tilde{\tilde{\phi}}(s)\big)\,ds \\ &+ A^{-\frac{\alpha}{2}}\int_0^t R\big(\tilde{\tilde{\phi}}(s)\big)\,ds - A^{-\frac{\alpha}{2}}\int_0^t f(s)\,ds = \int_0^t A^{-\frac{\alpha}{2}}G\big(\tilde{\tilde{\phi}}(s)\big)\,d\tilde{\tilde{W}}(s). \end{aligned}$$

However,

$$\int_0^t A^{-\frac{\alpha}{2}}G\big(\tilde{\tilde{\phi}}(s)\big)\,d\tilde{\tilde{W}}(s) = A^{-\frac{\alpha}{2}}\int_0^t G\big(\tilde{\tilde{\phi}}(s)\big)\,d\tilde{\tilde{W}}(s).$$

Thus $A^{-\frac{\alpha}{2}}\tilde{\tilde{\phi}}(t)$, $t\in[0,T]$ is continuous as a $D(A^\beta)$-valued process with $\beta := \frac{\alpha}{2}$. Hence

$$\begin{aligned} &\big(\tilde{\tilde{\phi}}(t)|\psi\big) \, - \big(\tilde{\tilde{\phi}}(0)|\psi\big) + \int_0^t \big\langle A\tilde{\tilde{\phi}}(s)|\psi\big\rangle\,ds + \int_0^t \big\langle B\big(\tilde{\tilde{\phi}}(s)\big)|\psi\big\rangle\,ds \\ &+ \int_0^t \big\langle R\big(\tilde{\tilde{\phi}}(s)\big)|\psi\big\rangle\,ds = \int_0^t \big\langle f(s)|\psi\big\rangle\,ds + \Big\langle \int_0^t G\big(\tilde{\tilde{\phi}}(s)\big)\,d\tilde{\tilde{W}}(s)\Big|\psi\Big\rangle, \\ &\qquad\qquad\qquad\qquad\qquad\qquad \forall\, t\in[0,T] \quad \forall\,\psi\in D(A^\beta). \end{aligned}$$

In conclusion, the conditions from Definition 4 hold with $\big(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\}_{t\geq 0}, \hat{\mathbb{P}}\big) = \big(\tilde{\tilde{\Omega}}, \tilde{\tilde{\mathcal{F}}}, \{\tilde{\tilde{\mathcal{F}}}_t\}_{t\geq 0}, \tilde{\tilde{\mathbb{P}}}\big)$, $\hat{W} = \tilde{\tilde{W}}$ and $\phi = \tilde{\tilde{\phi}}$. This completes the proof of Theorem 4.1.

5 Example

Let

$$G(\phi,\xi)(t,x) := \sum_{i=1}^{\infty}\big[\big(b^{(i)}(x)\cdot\nabla\big)\phi(t,x) + c^{(i)}(x)\phi(t,x)\big]\frac{d\beta^{(i)}(t)}{dt}, \tag{5.1}$$

where

$\beta^{(i)}, i\in\mathbb{N}$ - independent standard Brownian motions,
$b^{(i)} : \overline{D} \to \mathbb{R}^d$ - of class $\mathcal{C}^\infty$, $\quad i\in\mathbb{N}$
$c^{(i)} : \overline{D} \to \mathbb{R}$ - of class $\mathcal{C}^\infty$, $\quad i\in\mathbb{N}$

are given. Assume that

$$C_1 := \sum_{i=1}^{\infty} \big(\|b^{(i)}\|_{L^\infty}^2 + \|\mathrm{div} b^{(i)}\|_{L^\infty}^2 + \|c^{(i)}\|_{L^\infty}^2 \big) < \infty \tag{5.2}$$

and

$$\sum_{j,k=1}^{d} \big(2\delta_{jk} - \sum_{i=1}^{\infty} b_j^{(i)}(x) b_k^{(i)}(x) \big) \zeta_j \zeta_k \geq a|\zeta|^2, \qquad \zeta \in \mathbb{R}^d \tag{5.3}$$

for some $a \in (\frac{4}{3}, 2]$. Assumption (5.3) is equivalent to the following one

$$\sum_{i=1}^{\infty} \sum_{j,k=1}^{d} b_j^{(i)}(x) b_k^{(i)}(x) \big) \zeta_j \zeta_k \leq 2 \sum_{j,k=1}^{d} \delta_{jk} \zeta_j \zeta_k - a|\zeta|^2 = (2-a)|\zeta|^2. \tag{5.4}$$

Let $Y := l^2(\mathbb{N})$ and put

$$G(\phi)h = \sum_{i=1}^{\infty} \big[\big(b^{(i)} \cdot \nabla \big) \phi + c^{(i)} \phi \big] h_i, \qquad \phi \in V, \quad h = (h_i) \in l^2(\mathbb{N}).$$

We will show that G fulfils assumption **(A.3)**. Since the mapping G is linear, it is Lipschitz continuous provided that it is bounded. We will show that

$$2\langle A\phi | \phi \rangle - \|G(\phi)\|_{\mathcal{L}_{HS}(Y,H)}^2 \geq \eta \|\phi\|_V^2 - \lambda_0 |\phi|_H^2, \qquad \phi \in V \tag{$\tilde{\mathbf{G}}$}$$

for some constants λ_0 and $\eta \in (\frac{4}{3}, 2]$.

Moreover, G extends to a linear mapping $G : H \to \mathcal{L}_{HS}(Y, V_{-\gamma})$ for some $\gamma \geq 1$ and

$$\|G(\phi)\|_{\mathcal{L}_{HS}(Y,V_{-\gamma})}^2 \leq C|\phi|_H^2, \qquad \phi \in H. \tag{$\tilde{\mathbf{G}}*$}$$

for some $C > 0$.

Proof of ($\tilde{\mathbf{G}}$). Let us consider a standard orthonormal basis $h^{(i)} = (h_j^{(i)})$, $i \in \mathbb{N}$ in $l^2(\mathbb{N})$. Let $\phi \in V$. Then, for each $i \in \mathbb{N}$, we have

$$\begin{aligned} \big| G(\phi) h^{(i)} \big|_H^2 &= \left(\sum_{j=1}^{d} b_j^{(i)} \frac{\partial \phi}{\partial x_j} + c^{(i)} \phi \Big| \sum_{k=1}^{d} b_k^{(i)} \frac{\partial \phi}{\partial x_k} + c^{(i)} \phi \right)_H \\ &= \left(\sum_{j=1}^{d} b_j^{(i)} \frac{\partial \phi}{\partial x_j} \Big| \sum_{k=1}^{d} b_k^{(i)} \frac{\partial \phi}{\partial x_k} \right)_H + 2 \left(\sum_{j=1}^{d} b_j^{(i)} \frac{\partial \phi}{\partial x_j} \Big| c^{(i)} \phi \right)_H + \big| c^{(i)} \phi \big|_H^2. \end{aligned}$$

Thus

$$\|G(\phi)\|^2_{\mathcal{L}_{HS}(Y,H)} = \sum_{i=1}^{\infty} \big|G(\phi)h^{(i)}\big|^2_H = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{d} b_j^{(i)} \frac{\partial \phi}{\partial x_j} \Big| \sum_{k=1}^{d} b_k^{(i)} \frac{\partial \phi}{\partial x_k} \right)_H$$
$$+ 2 \sum_{i=1}^{\infty} \left(\sum_{j=1}^{d} b_j^{(i)} \frac{\partial \phi}{\partial x_j} \Big| c^{(i)} \phi \right)_H + \sum_{i=1}^{\infty} \big|c^{(i)}\phi\big|^2_H. \tag{5.5}$$

Let us estimate each term on the right-hand side. By (5.4)

$$\sum_{i=1}^{\infty} \left(\sum_{j=1}^{d} b_j^{(i)} \frac{\partial \phi}{\partial x_j} \Big| \sum_{k=1}^{d} b_k^{(i)} \frac{\partial \phi}{\partial x_k} \right)_H = \int_D \sum_{i=1}^{\infty} \sum_{j,k=1}^{d} b_j^{(i)}(x) b_k^{(i)}(x) \frac{\partial \phi}{\partial x_j} \frac{\partial \phi}{\partial x_k}\, dx$$
$$\leq (2-a)|\nabla \phi|^2_H, \qquad \phi \in V.$$

For each $i \in \mathbb{N}$ we have

$$\left(\sum_{j=1}^{d} b_j^{(i)} \frac{\partial \phi}{\partial x_j} \Big| c^{(i)} \phi \right)_H \leq \|b^{(i)}\|_{L^\infty} \|c^{(i)}\|_{L^\infty} \|\phi\|_V |\phi|_H$$
$$\leq \frac{1}{2} \big(\|b^{(i)}\|^2_{L^\infty} + \|c^{(i)}\|^2_{L^\infty} \big) \|\phi\|_V |\phi|_H, \quad \phi \in V.$$

Thus for any $\varepsilon > 0$

$$2 \sum_{i=1}^{\infty} \left(\sum_{j=1}^{d} b_j^{(i)} \frac{\partial \phi}{\partial x_j} \Big| c^{(i)} \phi \right)_H \leq \sum_{i=1}^{\infty} \big(\|b^{(i)}\|^2_{L^\infty} + \|c^{(i)}\|^2_{L^\infty} \big) \|\phi\|_V |\phi|_H$$
$$= C_1 \|\phi\|_V \cdot |\phi|_H \leq \varepsilon \|\phi\|^2_V + \frac{C_1^2}{4\varepsilon} |\phi|^2_H, \quad \phi \in V,$$

C_1 is defined by (5.2). The third term in (5.5) we estimate as follows

$$\sum_{i=1}^{\infty} \big|c^{(i)}\phi\big|^2_H \leq \sum_{i=1}^{\infty} \|c^{(i)}\|^2_{L^\infty} |\phi|^2_H = C_1 |\phi|^2_H, \qquad \phi \in V.$$

Hence

$$\|G(\phi)\|^2_{\mathcal{L}_{HS}(Y,H)} \leq (2+\varepsilon-a)\|\phi\|^2_V + \left(\frac{C_1^2}{4\varepsilon} + C_1 \right) |\phi|^2_H, \qquad \phi \in V$$

and

$$2\langle A\phi | \phi \rangle - \|G(\phi)\|^2_{\mathcal{L}_{HS}(Y,H)} \geq 2\|\phi\|^2_V - (2+\varepsilon-a)\|\phi\|^2_V - \left(\frac{C_1^2}{4\varepsilon} + C_1 \right) |\phi|^2_H$$
$$= (a-\varepsilon)\|\phi\|^2_V - \left(\frac{C_1^2}{4\varepsilon} + C_1 \right) |\phi|^2_H, \qquad \phi \in V.$$

It is sufficient to take $\varepsilon > 0$ such that $a - \varepsilon \in (\frac{4}{3}, 2]$. (Let us notice that such an ε exists.) Thus condition $(\tilde{\mathbf{G}})$ holds with $\eta := a - \varepsilon$ and $\lambda_0 := \frac{C_1^2}{4\varepsilon} + C_1$. $\square$

Proof of $(\tilde{\mathbf{G}}*)$. Let $b = (b_1, ..., b_d) : \overline{D} \to \mathbb{R}^d$. Then

$$\sum_{j=1}^{d} \frac{\partial}{\partial x_j}(b_j\phi) = \sum_{j=1}^{d}\left(\frac{\partial b_j}{\partial x_j}\phi + b_j\frac{\partial \phi}{\partial x_i}\right) = (\mathrm{div} b)\phi + \sum_{j=1}^{d} b_j\frac{\partial \phi}{\partial x_j}.$$

Thus for every $\psi \in V$

$$\begin{aligned}\int_D (\sum_{j=1}^{d} b_j\frac{\partial \phi}{\partial x_j})\psi\, dx &= \sum_{j=1}^{d}\int_D \frac{\partial}{\partial x_j}(b_j\phi)\psi\, dx - \int_D (\mathrm{div} b)\phi\psi\, dx \\ &= -\sum_{j=1}^{d}\int_D (b_j\phi)\frac{\partial \psi}{\partial x_j}\, dx - \int_D (\mathrm{div} b)\phi\psi\, dx.\end{aligned}$$

Hence using the Hőlder inequality, we obtain the following estimates

$$\begin{aligned}|\int_D (\sum_{j=1}^{d} b_j\frac{\partial \phi}{\partial x_j})\psi\, dx| &\le \left|\int_D \sum_{j=1}^{d} b_j\phi\frac{\partial \psi}{\partial x_j}\, dx\right| + \left|\int_D (\mathrm{div} b)\phi\psi\, dx\right| \\ &\le \|b\|_{L^\infty}|\phi|_H\|\psi\|_V + \|\mathrm{div} b\|_{L^\infty}|\phi|_H\|\psi\|_V.\end{aligned}$$

Thus

$$\|(b\cdot\nabla)\phi\|_{V'} \le \big(\|b\|_{L^\infty} + \|\mathrm{div} b\|_{L^\infty}\big)\cdot|\phi|_H.$$

Moreover,

$$\|c^{(i)}\phi\|_{V'} \le const\|c^{(i)}\|_{L^\infty}|\phi|_H.$$

Then $G(\phi)h^{(i)} = \big(b^{(i)}\cdot\nabla\big)\phi + c^{(i)}\phi$ and

$$|G(\phi)h^{(i)}|_{V'}^2 = |\big(b^{(i)}\cdot\nabla\big)\phi + c^{(i)}\phi|_{V'}^2 \le 2\big(|\big(b^{(i)}\cdot\nabla\big)\phi|_{V'}^2 + |c^{(i)}\phi|_{V'}^2\big).$$

Hence

$$\begin{aligned}\|G(\phi)h\|_{\mathcal{L}_{HS}(Y,V')}^2 &= \sum_{i=1}^{\infty}|G(\phi)h^{(i)}|_{V'}^2 \le 2\sum_{i=1}^{\infty}\Big(|\big(b^{(i)}\cdot\nabla\big)\phi|_{V'}^2 + |c^{(i)}\phi|_{V'}^2\Big) \\ &\le 2\sum_{i=1}^{\infty}\big(2\|b^{(i)}\|_{L^\infty}^2 + 2\|\mathrm{div} b^{(i)}\|_{L^\infty}^2 + \|c^{(i)}\|_{L^\infty}^2\big)|\phi|_H^2.\end{aligned}$$

Hence, $G(\phi) \in \mathcal{L}_{HS}(Y, V')$ and

$$\|G(\phi)\|_{\mathcal{L}_{HS}(Y,V')} \le C\cdot|\phi|_H,$$

where $C = 2C_1$. In conclusion, condition $(\tilde{\mathbf{G}}*)$ holds with $\gamma = 1$. $\square$

6 Appendix A

Let us recall the first part of assumption (A.3).

The mapping $G : V \to \mathcal{L}_{HS}(Y,H)$ is Lipschitz continuous and

$$2\big\langle A\phi|\phi\big\rangle - \|G(\phi)\|^2_{\mathcal{L}_{HS}(Y,H)} \geq \eta\|\phi\|^2_V - \lambda_0|\phi|^2_H - \rho, \qquad \phi \in V \tag{G}$$

for some constants λ_0, ρ and $\eta \in (0,2]$.

Since $\big\langle A\phi|\phi\big\rangle - \eta\|\phi\|^2_V = (2-\eta)\|\phi\|^2_V$, inequality (G) can be written in the following form

$$\|G(\phi)\|^2_{\mathcal{L}_{HS}(Y,H)} \leq 2\big\langle A\phi|\phi\big\rangle - \eta\|\phi\|^2_V + \lambda_0|\phi|^2_H + \rho, \qquad \phi \in V \tag{G'}$$

The following proof of Lemma 4.3 is standard, see [10]. However, we provide all details to indicate the importance of the assumption (4.2) on p.

Proof of estimates (4.3), (4.4) and (4.5) under the assumption (G).

Let p satisfy condition (4.2), i.e.

$$\begin{cases} p \in \big[2, 2+\frac{\eta}{2-\eta}\big) & \text{if} \quad \eta \in (0,2) \\ p \in [2,\infty) & \text{if} \quad \eta = 2. \end{cases}$$

We apply the Itô Lemma to the function $F(x) = |x|^p_H =: |x|^p$, $x \in H$. Since

$$\frac{\partial F}{\partial x} = p \cdot |x|^{p-2} \cdot x, \qquad \left\|\frac{\partial^2 F}{\partial x^2}\right\| \leq p(p-1) \cdot |x|^{p-2}, \qquad x \in H,$$

we infer that

$$\begin{aligned} d\big[|\phi_n(t)|^p\big] = &\Big[p\,|\phi_n(t)|^{p-2}\big\langle \phi_n(t)| - A\phi_n(t) - B_n\big(\phi_n(t)\big) - R(\phi_n(t)) + P_n f(t)\big\rangle \\ &+ \frac{1}{2}\ \mathrm{tr}\ \big[P_n G(\phi_n(t))\frac{\partial^2 F}{\partial x^2}\big(P_n G(\phi_n(t))\big)^*\big]\Big]\,dt \\ &+ p\,|\phi_n(t)|^{p-2}\big\langle \phi_n(t)|G(\phi_n(t))\,dW(t)\big\rangle \\ = &\Big[-p\,|\phi_n(t)|^{p-2}\|\phi_n(t)\|^2 - p\,|\phi_n(t)|^{p-2}\big\langle \phi_n(t)|R(\phi_n(t))\big\rangle \\ &+ p\,|\phi_n(t)|^{p-2}\big\langle \phi_n(t)|P_n f(t)\big\rangle \\ &+ \frac{1}{2}\ \mathrm{tr}\ \big[P_n G(\phi_n(t))\frac{\partial^2 F}{\partial x^2}\big(P_n G(\phi_n(t))\big)^*\big]\Big]\,dt \\ &+ p\,|\phi_n(t)|^{p-2}\big\langle \phi_n(t)|G(\phi_n(t))\,dW(t)\big\rangle, \qquad t \geq 0. \end{aligned}$$

Thus

$$\begin{aligned} d\big[|\phi_n(t)|^p\big] + p\,|\phi_n(t)|^{p-2}\|\phi_n(t)\|^2 \ \le\ & -p\,|\phi_n(t)|^{p-2}\big\langle \phi_n(t)|R(\phi_n(t))\big\rangle\,dt \\ & + p\,|\phi_n(t)|^{p-2}\big\langle \phi_n(t)|f(t)\big\rangle\,dt \\ & + \frac{1}{2}p(p-1)\,|\phi_n(t)|^{p-2}\cdot\|P_nG(\phi_n(t))\|^2_{\mathcal{L}_{HS}(Y,H)}\,dt \\ & + p\,|\phi_n(t)|^{p-2}\big\langle \phi_n(t)|G(\phi_n(t))\,dW(t)\big\rangle, \qquad t\ge 0. \end{aligned}$$

By (G') we have

$$\|P_nG(\phi_n(t))\|^2_{\mathcal{L}_{HS}(Y,H)} \le (2-\eta)\,\|\phi_n(t)\|^2 + \lambda_0|\phi_n(t)|^2 + \rho, \qquad t\ge 0.$$

By Lemma 3.2 (2)

$$-\big\langle \phi_n(t)|R(\phi_n(t))\big\rangle \le |\phi_n(t)|^2, \qquad t\ge 0.$$

Moreover, by the Schwarz inequality for any $\varepsilon>0$ we get

$$\begin{aligned} \big\langle f(t)|\phi_n(t)\big\rangle \le\ & |f(t)|_{V'}\cdot\|\phi_n(t)\| = \frac{1}{(2\varepsilon)^{\frac12}}|f(t)|_{V'}\cdot(2\varepsilon)^{\frac12}\|\phi_n(t)\| \\ \le\ & \frac12\left(\frac{1}{2\varepsilon}|f(t)|^2_{V'} + 2\varepsilon\,\|\phi_n(t)\|^2\right) = \frac{1}{4\varepsilon}|f(t)|^2_{V'} + \varepsilon\,\|\phi_n(t)\|^2, \quad t\ge 0. \end{aligned}$$

Hence for $t\ge 0$

$$\begin{aligned} d\big[|\phi_n(t)|^p\big] &+ p\,|\phi_n(t)|^{p-2}\|\phi_n(t)\|^2\,dt \ \le p|\phi_n(t)|^p\,dt \\ &+ p|\phi_n(t)|^{p-2}\frac{1}{4\varepsilon}|f(t)|^2_{V'} + p|\phi_n(t)|^{p-2}\varepsilon\,\|\phi_n(t)\|^2\,dt \\ &+ \frac12 p(p-1)\,|\phi_n(t)|^{p-2}\cdot\Big[(2-\eta)\,\|\phi_n(t)\|^2 + \lambda_0|\phi_n(t)|^2 + \rho\Big]\,dt \\ &+ p\,|\phi_n(t)|^{p-2}\big\langle \phi_n(t)|G(\phi_n(t))\,dW(t)\big\rangle \end{aligned}$$

and therefore

$$\begin{aligned} d\big[|\phi_n(t)|^p\big] &+ \big[p - p\varepsilon - \frac12 p(p-1)(2-\eta)\big]\,|\phi_n(t)|^{p-2}\|\phi_n(t)\|^2 \\ &\le p|\phi_n(t)|^p\,dt + p\,|\phi_n(t)|^{p-2}\cdot\frac{1}{4\varepsilon}|f(t)|^2_{V'}\,dt \\ &\quad + \frac12 p(p-1)\,|\phi_n(t)|^{p-2}\cdot\big[\lambda_0|\phi_n(t)|^2 + \rho\big]\,dt \\ &\quad + p\,|\phi_n(t)|^{p-2}\big\langle \phi_n(t)|G(\phi_n(t))\,dW(t)\big\rangle. \end{aligned}$$

Moreover,

$$
\begin{aligned}
& p\,|\phi_n(t)|^{p-2}\cdot\frac{1}{4\varepsilon}|f(t)|^2_{V'}\;+\frac{1}{2}p(p-1)\,|\phi_n(t)|^{p-2}\cdot\big[\lambda_0|\phi_n(t)|^2+\rho\big]\\
&=|\phi_n(t)|^{p-2}\cdot\left[\frac{1}{2}p(p-1)\lambda_0\,|\phi_n(t)|^2+\frac{p}{4\varepsilon}|f(t)|^2_{V'}+\frac{1}{2}p(p-1)\rho\right]\\
&\le\left[\left(\frac{1}{2}p(p-1)\lambda_0\;+\varepsilon|f(t)|^2_{V'}\right)|\phi_n(t)|^p+C(\varepsilon,p,\rho)\big(|f(t)|^2_{V'}+1\big)\right],\quad t\ge 0.
\end{aligned}
$$

Thus we obtain

$$
\begin{aligned}
|\phi_n(t)|^p+&\big[p-p\varepsilon-\frac{1}{2}p(p-1)(2-\eta)\,\big]\int_0^t|\phi_n(s)|^{p-2}\|\phi_n(s)\|^2\,ds\\
&\le|\phi_n(0)|^p+\int_0^t\left(p+\frac{1}{2}p(p-1)\lambda_0\;+\varepsilon|f(s)|^2_{V'}\right)|\phi_n(s)|^p\,ds\\
&\quad+C(\varepsilon,p,\rho)\int_0^t\big(|f(s)|^2_{V'}+1\big)\,ds\\
&\quad+p\int_0^t|\phi_n(s)|^{p-2}\big\langle\phi_n(s)|G(\phi_n(s))\,dW(s)\big\rangle,\quad t\in[0,T].
\end{aligned}
\tag{6.1}
$$

Let us choose $\varepsilon>0$ such that $p-p\varepsilon-\frac{1}{2}p(p-1)(2-\eta)>0$, or equivalently,

$$
\varepsilon<1-\frac{1}{2}(p-1)(2-\eta).
$$

Observe that under condition (4.2) such an ε exists.

By Lemma 4.2, $\mathbb{E}[\int_0^T|\phi_n(s)|^q_H\,ds]<\infty$ for any $q\in[1,\infty)$. Hence by (G) and the equivalence of all norms in the finite dimensional space, we infer that the process

$$
\mu_n(t):=\int_0^t|\phi_n(s)|^{p-2}\big\langle\phi_n(s)|G(\phi_n(s))\,dW(s)\big\rangle,\qquad t\in[0,T]
$$

is a martingale and that $\mathbb{E}[\mu_n(t)]=0$. Thus we have

$$
\begin{aligned}
\mathbb{E}\big[|\phi_n(t)|^p\big]\;\le\;&\mathbb{E}\big[|\phi_n(0)|^p\big]\\
&+\int_0^t\left(p+\frac{1}{2}p(p-1)\lambda_0\;+\varepsilon|f(s)|^2_{V'}\right)\mathbb{E}\big[|\phi_n(s)|^p\big]\,ds\\
&+C(\varepsilon,p,\rho)\int_0^t\big(|f(s)|^2_{V'}+1\big)\,ds\quad\forall\,t\in[0,T].
\end{aligned}
\tag{6.2}
$$

Hence by the Gronwall Lemma there exists a constant $C>0$ such that

$$
\mathbb{E}\big[|\phi_n(t)|^p\big]\le C\qquad\forall\,t\in[0,T]\quad\forall\,n\ge 1.
\tag{6.3}
$$

Using this estimate in inequality (6.1), we also obtain

$$\mathbb{E}\Big[\int_0^T |\phi_n(s)|^{p-2}\|\phi_n(s)\|^2\,ds\Big] \le C_2(p), \qquad n\ge 1 \tag{6.4}$$

for some constant $C_2(p) > 0$. This completes the proof of inequalities (4.4) and (4.5).

By the Burkholder-Davis-Gundy inequality, see [18], and estimates (6.3)and (6.4) we obtain the following estimates

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{0\le s\le t}\Big|\int_0^s p\,|\phi_n(\sigma)|^{p-2}\big\langle\phi_n(\sigma)|P_nG(\phi_n(\sigma))\,dW(\sigma)\big\rangle\Big|\Big]\\
&\quad\le C\,p\cdot\mathbb{E}\Big[\Big(\int_0^t \big\||\phi_n(\sigma)|^{p-2}\big(\phi_n(\sigma)|\cdot\big)\,P_nG(\phi_n(\sigma))\big\|^2_{\mathcal{L}_{HS}(Y,H)}\,d\sigma\Big)^{\frac12}\Big]\\
&\quad= C\,p\cdot\mathbb{E}\Big[\Big(\int_0^t |\phi_n(\sigma)|^{2p-4}\big\|\big\langle\phi_n(\sigma)|\cdot\big\rangle\,P_nG(\phi_n(\sigma))\big\|^2_{\mathcal{L}_{HS}(Y,H)}\,d\sigma\Big)^{\frac12}\Big]\\
&\quad\le C\,p\cdot\mathbb{E}\Big[\Big(\int_0^t |\phi_n(\sigma)|^{2p-4}\cdot|\phi_n(\sigma)|^2\cdot\|G(\phi_n(\sigma))\|^2_{\mathcal{L}_{HS}(Y,H)}\,d\sigma\Big)^{\frac12}\Big]\\
&\quad= C\,p\cdot\mathbb{E}\Big[\Big(\int_0^t |\phi_n(\sigma)|^{2p-2}\cdot\|G(\phi_n(\sigma))\|^2_{\mathcal{L}_{HS}(Y,H)}\,d\sigma\Big)^{\frac12}\Big]\\
&\quad\le C\,p\cdot\mathbb{E}\Big[\sup_{0\le\sigma\le t}|\phi_n(\sigma)|^{\frac p2}\Big(\int_0^t |\phi_n(\sigma)|^{p-2}\cdot\|G(\phi_n(\sigma))\|^2_{\mathcal{L}_{HS}(Y,H)}\,d\sigma\Big)^{\frac12}\Big]\\
&\quad\le C\,p\cdot\mathbb{E}\Big[\sup_{0\le\sigma\le t}|\phi_n(\sigma)|^{\frac p2}\Big(\int_0^t |\phi_n(\sigma)|^{p-2}\cdot\big[\lambda_0\,|\phi_n(\sigma)|^2+\rho\\
&\qquad+(2-\eta)\|\phi_n(\sigma)\|^2\big]\,d\sigma\Big)^{\frac12}\Big]\\
&\quad\le \frac12\,\mathbb{E}\big[\sup_{0\le\sigma\le t}|\phi_n(\sigma)|^p\big]+\frac12 C^2p^2\,\mathbb{E}\Big[\int_0^t |\phi_n(\sigma)|^{p-2}\cdot\big[\lambda_0\,|\phi_n(\sigma)|^2+\rho\\
&\qquad+(2-\eta)\|\phi_n(\sigma)\|^2\big]\,d\sigma\Big]\\
&\quad=\frac12\,\mathbb{E}\big[\sup_{0\le s\le t}|\phi_n(s)|^p\big]+\frac12 C^2p^2\cdot\mathbb{E}\Big[\int_0^t \lambda_0\,|\phi_n(s)|^p\,ds\Big]\\
&\qquad+\frac12 C^2p^2\rho\cdot\mathbb{E}\Big[\int_0^t |\phi_n(s)|^{p-2}\,ds\Big]\\
&\qquad+\frac12 C^2p^2(2-\eta)\cdot\mathbb{E}\Big[\int_0^t |\phi_n(s)|^{p-2}\|\phi_n(s)\|^2\,ds\Big]
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}\,\mathbb{E}\Big[\sup_{0\le s\le t}|\phi_n(s)|^p\Big] + \frac{1}{2}C^2p^2\lambda_0\ \cdot \mathbb{E}\bigg[\int_0^t \sup_{0\le s\le \sigma}|\phi_n(s)|^p\,d\sigma\bigg]\\
&\quad + \frac{1}{2}C^2p^2\rho\cdot\mathbb{E}\bigg[\int_0^t |\phi_n(s)|^{p-2}\,ds\bigg]\\
&\quad + \frac{1}{2}C^2p^2(2-\eta)\cdot\mathbb{E}\bigg[\int_0^t |\phi_n(s)|^{p-2}\|\phi_n(s)\|^2\,d\sigma\bigg]\\
&\le \frac{1}{2}\,\mathbb{E}\Big[\sup_{0\le s\le t}|\phi_n(s)|^p\Big] + \frac{1}{2}C^2p^2\lambda_0\ \cdot \mathbb{E}\bigg[\int_0^t \sup_{0\le s\le \sigma}|\phi_n(s)|^p\,d\sigma\bigg] + const.
\end{aligned}
$$

Thus by (6.2) we have

$$
\begin{aligned}
\mathbb{E}\Big[\sup_{0\le s\le t}|\phi_n(s)|^p\Big] &\le \mathbb{E}\big[|\phi_n(0)|^p\big]\\
&\quad + \int_0^t \Big(p+\frac{1}{2}p(p-1)\lambda_0\ +\varepsilon|f|_{V'}^2\Big)\ \mathbb{E}\Big[\sup_{0\le r\le s}|\phi_n(r)|^p\Big]\,ds\\
&\quad + C(\varepsilon,p,\rho)\int_0^t \big(|f(s)|_{V'}^2+1\big)\,ds + \frac{1}{2}\,\mathbb{E}\Big[\sup_{0\le s\le t}|\phi_n(s)|^p\Big]\\
&\quad + \frac{1}{2}C^2p^2\lambda_0\ \cdot \int_0^t \mathbb{E}\Big[\sup_{0\le s\le\sigma}|\phi_n(s)|^p\Big]d\sigma + const.
\end{aligned}
$$

Hence by the Gronwall Lemma, we get (4.3), i.e.

$$
\mathbb{E}\Big[\sup_{0\le s\le t}|\phi_n(s)|^p\Big] \le C_1(p)
$$

for some constant $C_1(p) > 0$. This completes the proof of estimates (4.3), (4.4) and (4.5) in Lemma 4.3. □

7 Appendix B

Lemma 7.1. *Let $\beta > d+2$ and let $\phi_n \to \phi$ in $L^2(0,T;H)$. Then for all $t \in [0,T]$ and all $\psi \in D(A^{\frac{\beta}{2}})$ we have*

$$
\lim_{n\to\infty}\Big\langle \int_0^t P_nB\big(\phi_n(s)\big)\,ds\big|\psi\Big\rangle = \Big\langle \int_0^t B\big(\phi(s)\big)\,ds\big|\psi\Big\rangle. \tag{7.1}
$$

Proof. We have

$$
\begin{aligned}
&\Big\langle \int_0^t P_nB\big(\phi_n(s)\big)\,ds\big|\psi\Big\rangle = \Big\langle \int_0^t B\big(\phi_n(s)\big)\,ds\big|P_n\psi\Big\rangle\\
&= \Big\langle \int_0^t B\big(\phi_n(s)\big)\,ds\big|P_n\psi-\psi\Big\rangle + \Big\langle \int_0^t B\big(\phi_n(s)\big)\,ds\big|\psi\Big\rangle =: I_1(n)+I_2(n).
\end{aligned}
$$

For the first term on the right hand side, we have the following estimate

$$\begin{aligned}|I_1(n)| &= \left|\left\langle \int_0^t B\big(\phi_n(s)\big)\,ds \big| P_n\psi - \psi \right\rangle\right| \\ &\le \int_0^t \big|B\big(\phi_n(s)\big)\big|_{D(A^{-\frac{\beta}{2}})}\,ds \cdot \|P_n\psi - \psi\|_{D(A^{\frac{\beta}{2}})} \\ &\le \int_0^T |\phi_n(s)|_H^2\,ds \cdot \|P_n\psi - \psi\|_{D(A^{\frac{\beta}{2}})}.\end{aligned}$$

Since $P_n \to I$ strongly in $D(A^{\frac{\beta}{2}})$ as $n \to \infty$ and $(\phi_n)_{n\ge 1}$ is bounded in $L^2(0,T;H)$, we infer that $\lim_{n\to\infty} I_1(n) = 0$.

Let us move to the second term. Let us denote $\phi_n = (u_n, \vartheta_n)$, $\phi = (u, \vartheta)$. Then

$$\begin{aligned}B(\phi_n) - B(\phi) &= \big(B_1(u_n,u_n), B_2(u_n,\vartheta_n)\big) - \big(B_1(u,u), B_2(u,\vartheta)\big) \\ &= \big(B_1(u_n-u,u_n) + B_1(u,u_n-u), B_2(u_n-u,\vartheta_n) + B_2(u,\vartheta_n-\vartheta)\big).\end{aligned}$$

Thus using the estimates (3.6) and (3.12), we obtain

$$\begin{aligned}&\left|\left\langle \int_0^t B\big(\phi_n(s)\big)\,ds \big| \psi \right\rangle - \left\langle \int_0^t B\big(\phi(s)\big)\,ds \big| \psi \right\rangle\right| \\ &\le C \cdot \|\phi_n - \phi\|_{L^2(0,T;H)} \big(\|\phi_n\|_{L^2(0,T;H)} + \|\phi\|_{L^2(0,T;H)}\big) \|\psi\|_{D(A^{\frac{\beta}{2}})} \to 0\end{aligned}$$

as $n \to \infty$ (C stands for some positive constant). Thus

$$\lim_{n\to\infty} I_2(n) = \left\langle \int_0^t B\big(\phi(s)\big)\,ds \big| \psi \right\rangle,$$

which completes the proof of Lemma. □

References

[1] R. Adams, “Sobolev spaces”, Academic Press, 1975.

[2] D. Aldous, “Stopping times and tightness”, Ann. of Prob., Vol.6, No. 2, pp. 335-340, 1978.

[3] P. Billingsley, “Convergence of probability measures”, Wiley, New York, 1969.

[4] H. Brezis, “Analyse fonctionnelle”, Masson, 1983.

[5] Z. Brzeźniak, M. Capiński, F. Flandoli, "Stochastic partial differential equations and turbulence", Mathematical Models and Methods in Applied Sciences, Vol.1, No. 1, pp. 41-59, 1991.

[6] Z. Brzeźniak, Y. Li, "Asymptotic compactness and absorbing sets for 2D stochastic Navier-Stokes equations on some unbounded domains", Trans. Am. Math. Soc., Vol. 358, pp. 5587-5629, 2006.

[7] G. Da Prato, J. Zabczyk, "Stochastic Equations in Infinite Dimensions", Cambridge University Press, 1992.

[8] J. Duan, A. Millet, "Large deviations for the Boussinesq equations under random influences", Stochastic processes and their Applications, Vol. 119, No. 6, pp. 2052-2081, 2009.

[9] B. Ferrario, "The Bénard problem with random perturbations: dissipativity and invariant measures", NoDEA, No. 4, pp. 101-121, 1997.

[10] F. Flandoli, D. Gątarek, "Martingale and stationary solutions for stochastic Navier-Stokes equations", Prob. Theory Related Fields, Vol. 102, No. 3, pp. 367-391, 1995.

[11] C. Foiaş, O. Manley, R. Temam, "Atractors for the Bénard Problem: Existence and physical bounds on their fractal dimension", Nonlinear Analysis, Vol. 11, pp. 939-967, 1987.

[12] J.M. Ghidaglia, "On the fractal dimension of attractors for viscous incompressible fluid flows", SIAM J. Math. Anal., Vol 17, No. 5, pp. 1139-1157, 1986.

[13] N.V. Krylov, "Introduction to the theory of diffusion processes", American Mathematical Society, 1995.

[14] J.L. Lions, "Quelques méthodes de résolution des problèmes aux limites non linéaires", Dunod, Paris, 1969.

[15] M. Métivier, "Stochastic partial differential equations in infinite dimensional spaces", Scuola Normale Superiore, Pisa 1988.

[16] M. Métivier, Semimartingales, Gruyter, 1982.

[17] R. Mikulevicius, B.L. Rozovskii, "Global L_2-solutions of stochastic Navier-Stokes equations", Ann. of Prob., Vol. 33, No.1, pp. 137-176, 2005.

[18] D. Revuz, M. Yor, "Continuous martingales and Brownian motion", Springer-Verlag, 1999.

[19] W.A. Strauss, "On continuity of functions with values in various Banach spaces", Pacific J. Math., Vol. 19, No. 3, pp. 543-555, 1966.

[20] R. Temam, "Navier-Stokes equations. Theory and numerical analysis", North Holland Publishing Company, Amsterdam - New York - Oxford, 1979.

[21] R. Temam, R., "Infinite-dimensional dynamical systems in mechanics and physics", Springer-Verlag, 1988.

[22] R. Temam, "Navier-Stokes equations and nonlinear functional analysis", SIAM, Philadelphia, Pensylvania, 1995.

[23] M.J. Vishik, A.V. Fursikov, "Mathematical Problems of Statistical Hydromechanics", Kluwer Academic Publishers, Dordrecht, 1988.