Global and Stochastic Analysis
Vol. 1, No. 2, December 2011

The probabilistic representation of the exponent of a class of pseudo-differential operators

S.Albeverio[a], N.Smorodina[b] and M.Faddeev[c]

[a]IAM, Univ. Bonn, D 53115 Bonn, Germany;
BiBoS Research Centre,
D 33615 Bielefeld, Germany.
e-mail: albeverio@uni-bonn.de

[b]St.-Petersburg State University,
St.-Petersburg, Russia.
e-mail: smorodin@ns2691.spb.edu

[c]St.-Petersburg State University,
St.-Petersburg, Russia.
e-mail: mmf@ns2691.spb.edu

Received by the Editorial Board on September 19, 2011

Abstract

In the present paper we construct a probabilistic representation of the operator exponent $e^{t\mathcal{A}}$ where $\mathcal{A}$ belongs to some class of pseudo-differential operators.

Keywords: Poisson random measures, evolution equation, limit theorems, pseudo-differential operator

2010 Mathematics subject classification: 28C20 60H05 60G57

1. Introduction.

We consider an evolution equation

$$\frac{\partial u}{\partial t} = \mathcal{A}u, \tag{1}$$

where $\mathcal{A}$ is a linear operator. Given an operator $\mathcal{A}$ let $e^{t\mathcal{A}},\ t \geq 0$ denote the operator exponent that is a family of linear operators such that for every $t > 0$ the operator $e^{t\mathcal{A}}$ maps the real or complex-valued function $\varphi(x), x \in \mathbb{R}$ into a solution $u(t,x)$ of the Cauchy problem of the equation (1) with an initial condition $u(0,x) = \varphi(x)$.

It is well known (see [15]), that the exponents of some pseudo-differential operators and the exponent of the operator $\frac{d^2}{dx^2}$ have probabilistic representations. Namely, for the operator $\frac{d^2}{dx^2}$ we have

$$e^{t\frac{1}{2}\frac{d^2}{dx^2}}\varphi(x) = \mathbb{E}\varphi(x + w(t)),$$

where $w(t)$ is a standard Wiener process, $w(0) = 0$, and for the operator $\mathcal{A}$, that acts as

$$\mathcal{A}\varphi(x) = \int_{\mathbb{R}} (\varphi(x+y) - \varphi(x))\Lambda(dy),$$

if the measure Λ satisfies the condition $\int_{\mathbb{R}} \min(|x|, 1)\Lambda(dx) < \infty$, and as

$$\mathcal{A}\varphi(x) = \int_{\mathbb{R}} (\varphi(x+y) - \varphi(x) - y\varphi'(x)\mathbf{1}_{[-1,1]}(y))\Lambda(dy), \tag{2}$$

if the measure Λ satisfies the condition $\int_{\mathbb{R}} \min(|x|^2, 1)\Lambda(dx) < \infty$, we have

$$e^{t\mathcal{A}}\varphi(x) = \mathbb{E}\varphi(x + \xi(t)) \tag{3}$$

where $\xi(t)$ is a jump Lévy process with the Lévy measure Λ, $\xi(0) = 0$.

Note that if the operator exponent can be represented in the form (3) the fundamental solution $q(t,x,y)$ of the equation (1) for every $t > 0$ coincides with one-dimensional distribution of the process $x + \xi(t)$.

It is important to note that the representation (3) can be directly generalized neither for higher order differential operators nor for operators that look like (2) but include higher order derivatives of φ. The simplest explanation of this fact is based on the maximum principle. Namely it follows from (3) that the operator $\mathcal{A}$ satisfies the following property: if the function φ has a maximum at point x, then $\mathcal{A}\varphi(x) \leq 0$. It is clear that higher order differential operators do not satisfy this property.

An analog of the representation (3) was considered in a number of papers (see [9, 6, 14, 3]). In this representation instead of usual probability processes so-called pseudo-processes were used.

A concept of a pseudo-process appeared for the first time in a paper of Yu. Daletski (see. [5]). Note that a pseudo-process is not really a stochastic process. Actually it is defined by a fundamental solution of an equation

$$\frac{\partial u}{\partial t} = k_n \frac{\partial^n u}{\partial x^n}.$$

It is easy to prove that pseudo-processes exist only in a sense of finite-dimensional distributions and do not generate any measure in the space of trajectories.

Nevertheless there are a lot of papers (see [14, 3]) concerning the properties of pseudo-processes. It was proved that a lot of functionals of the trajectories of the pseudo-processes are well-defined in the sense that they generate a signed measure on $\mathbb{R}$. Mention here that an analog of the arc-sine law [9, 11], of the central limit theorem [10] and of the Ito formula and the Ito stochastic calculus [9] has been proved for the pseudo-processes. We also mention that an analog of the Feynman-Kac formula has been proved in [13, 11].

For the exponent of the fourth order differential operator (case $n = 4$) an analog of the representation (3) were proposed in [6, 4, 7].

Further, in [16] stable measures with the index of stability greater than 2 were studied. Such measures are signed ones and hence they are not probability measures. Nevertheless for this class of measures an analogue of the Lévy-Khinchin representation was constructed. It was shown that in some sense these signed measures are limit measures for sums of independent random variables. It was also shown that these limit measures give us information about large deviations of sums of independent random variables.

The present paper provides a further development of this approach. Namely, instead of one-dimensional distributions we consider corresponding processes with independent increments.

We construct, in particular, a probabilistic representation of the operator exponent $e^{t\mathcal{A}}$, where $\mathcal{A}$ belongs to a class of pseudo-differential operators.

First we describe this class of operators.

Let g be a generalized function on $\mathbb{R}$ (see [8]), such that for every $\varepsilon > 0$ the restriction of g on $\mathbb{R}_\varepsilon = \mathbb{R} \setminus (-\varepsilon, \varepsilon)$ is a finite signed measure that is

$$|g|(\mathbb{R}_\varepsilon) < \infty,$$

and at the point 0 the generalized function g can have a singularity of a finite order. Namely, we suppose that for every $a > 0$ and some $r \in \mathbb{N}$ the generalized function g is continuous with respect to the norm $\|\cdot\|_{a,r}$, that is

$$|(g, \varphi)| \leq C_{a,r} \|\varphi\|_{a,r}, \tag{4}$$

where

$$\|\varphi\|_{a,r} = \sup_{|x|>a} |\varphi(x)| + \max_{0\leq j\leq r} \sup_{|x|\leq a} |\varphi^{(j)}(x)|.$$

We also suppose that

$$(g, \varphi) = 0 \tag{5}$$

if $\varphi \equiv C = \text{const}$.

As examples of such generalized functions g one can consider derivatives of the Dirac $\delta-$function and generalized functions that have the form of a regularization of a Borel function g, that satisfies the condition

$$\int_{\mathbb{R}} \min(|x|^r, 1)|g(x)|dx < \infty$$

for some $r \in \mathbb{N}$.

Note that if supp $g = \{0\}$ then for $r > 0$ and a constant $C_r > 0$

$$|(g, \varphi)| \leq C_r \max_{0 \leq j \leq r} |\varphi^{(j)}(0)|.$$

For a space of test functions we choose the space of all bounded infinitely differentiable functions with bounded derivatives of an arbitrary order.

For every generalized function g we construct an operator $\mathcal{A}_g$, by

$$\mathcal{A}_g f(x) = (g_y, f(x+y)), \tag{6}$$

where we denote the action of g on f with respect to the variable y by g_y.

We also consider operators $\mathcal{A}_g^c$ defined by

$$\mathcal{A}_g^c f(x) = (g_y, f(x+cy)), \tag{7}$$

where c is a constant (real or complex). Note, that we always suppose that f is defined on $\mathbb{R}$ and in (7) for complex c we understand $f(x+cy)$ as a substitution of $x+cy$ into an analytic continuation of f (assumed to exist).

To construct a probabilistic representation of the operator exponent of $\mathcal{A}_g$ we consider two objects defined by the generalized function g. The first object is a probability space $(\Omega, \mathcal{F}, P_g)$. Next we consider a subset Ω^0 of Ω (in all interesting cases $P_g(\Omega^0) = 0$) and on Ω^0 instead of a probability measure we define a generalized function L_g So our second object will be a triple $(\Omega^0, \mathcal{G}, L_g)$, where $\mathcal{G}$ is a set of test functions of L_g.

All random processes we define on the space Ω^0, and in the classical representation (3) instead of the mathematical expectation we use the generalized function L_g (for the same functional). Then we study the connection between the probability measure P_g and the generalized function L_g.

2. The space $(\Omega^0, \mathcal{G}, L_g)$.

Let Ω^0 denote the space of all discrete signed measures on $[0, T]$ with a finite spectrum (finite number of atoms). Each element ν of this space can be represented in the form $\nu = \sum_{k=1}^n x_k \delta_{t_k}$, where δ_{t_k} denotes a unit mass ($\delta-$measure) at a point t_k. We suppose that in this representation $|x_k| > 0$ for all k and all the points $t_k \in [0,T]$ are different. On Ω^0 we consider a topology generated by a total variation norm.

Let $f:\Omega^0 \to \mathbb{C}$ be a Borel function, for every $k \in \mathbb{N}_0,\ \mathbb{N}_0 = \{0,1,2\dots\}$; we use a notation f_k for a symmetric function of k two-dimensional variables defined by

$$f_k((t_1,x_1),(t_2,x_2),\dots,(t_k,x_k)) = f(\sum_{j=1}^{k} x_j\delta_{t_j}),\ t_j \in [0,T], x_j \in \mathbb{R}. \qquad (8).$$

Using (8) we get the following relations

$$f_{k+1}((t_1,x_1),\dots,(t_k,x_k),(t_{k+1},0)) = f_k((t_1,x_1),\dots,(t_k,x_k)) \qquad (9)$$

and

$$f_{k+1}((t_1,x_1),\dots,(t_k,x_k),(t_k,y_k)) = f_k((t_1,x_1),\dots,(t_k,x_k+y_k)),\ y_k \in \mathbb{R}.$$

In the next step, given a generalized function g on $\mathbb{R}$, we define a generalized function L_g on Ω^0.

First we define a space of test functions. To this end for every $k \in \mathbb{N} = \{1,2,3\dots\}, r \geq 1, a > 0$ we define a norm $\|\cdot\|_{k,r,a}$, by

$$\|h\|_{k,r,a} = \sum_{I\subset\{1,\dots,k\}} \sup_{t_1,\dots,t_k} \max_{p,p_i\leq r} \sup_{\substack{|x_i|\leq a,\\ i\in I}} \sup_{\substack{|x_i|>a,\\ i\notin I}} |\mathcal{D}_{I,p}h|, \qquad (10)$$

on the set of functions $h:\big([0,T]\times\mathbb{R}\big)^k \to \mathbb{R}$. Here $\mathcal{D}_{I,p}$ denotes the differential operator

$$\mathcal{D}_{I,p} = \prod_{i\in I} \frac{\partial^{p_i}}{\partial x_i^{p_i}}, \qquad (11)$$

for $I \subset \{1,\dots,r\},\ p = (p_1,\dots,p_k)$ and we suppose that the operator $\mathcal{D}_{I,p}$ acts only with respect to the variables $x_i, i \in I$

Note that in the case supp $g = \{0\}$ we use a simpler norm

$$\|h\|_{k,r} = \sup_{t_1,\dots,t_k} \max_{p,p_i\leq r} |\mathcal{D}_{\{1,\dots,k\},p}h((t_1,0),\dots,(t_k,0))| \qquad (12)$$

instead of (10).

As a space of test functions we use the set $\mathcal{G}$ of Borel functions on Ω^0, such that for every $k \in \mathbb{N}$ the function $f_k = f_k((t_1,x_1),\dots,(t_k,x_k))$ is infinitely differentiable with respect to x_i, and for every $r > 0, a > 0, M > 0$ the following series

$$\sum_{k=0}^{\infty} \frac{M^k\|f_k\|_{k,r,a}}{k!}, \qquad (13)$$

converges or if supp $g = \{0\}$ the series

$$\sum_{k=0}^{\infty} \frac{M^k \|f_k\|_{k,r}}{k!} \tag{14}$$

converges . In the latter case we denote the corresponding class by $\mathcal{G}_0$.

We say that the sequence of test functions $f^{(n)}$ converges to a test function f if $f_k^{(n)}((t_1, x_1), \dots, (t_k, x_k)) \underset{n\to\infty}{\longrightarrow} f_k((t_1, x_1), \dots, (t_k, x_k))$ for every k uniformly with respect to $t_i \in [0, T]$ and $x_i \in \mathbb{R}$ with all its derivatives (w.r.t. x_i) and for every $r > 0, a > 0, M > 0$ the following sums are bounded

$$\sum_{k=0}^{\infty} \frac{M^k \|f_k^{(n)}\|_{k,r,a}}{k!}$$

with respect to n.

Now under the class of test functions $\mathcal{G}$ (or $\mathcal{G}_0$, if supp $g = \{0\}$) we define a generalized function L_g. For every $f \in \mathcal{G}$ we put

$$(L_g, f) = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{[0,T]^k} (g^{\otimes k}, f_k) dm^k. \tag{15}$$

In (15) we suppose that the generalized function $g^{\otimes k}$ acts with respect to $x_1, \dots, x_k$, so that

$$(g^{\otimes k}, f_k)(t_1, \dots, t_k) = (g_{x_1} \times \dots \times g_{x_k}, f_k((t_1, x_1), \dots, (t_k, x_k))),$$

m^k denotes the Lebesgue measure on $[0, T]^k$ and we integrate with respect to $(t_1, \dots, t_k)$.

It follows from (4) that for some constant $C = C(a, r)$ and for every $k \in \mathbb{N}$ we have

$$\sup_{(t_1,\dots,t_k)} |(g^{\otimes k}, f_k)(t_1, \dots, t_k)| \le C^k \|f_k\|_{k,r,a}. \tag{16}$$

The convergence of the series (15) follows from (14) and (16).

For (L_g, f) we use a notation $L_g f$.

Below we consider not only the interval $[0, T]$, but also its subintervals. Now we introduce corresponding definitions.

Let $[t, s) \subset [0, T]$. By $\Omega^0_{t,s}$ we denote the set of all discrete signed measures with finite spectrum on the interval $[t, s)$. For every $u \in (t, s)$ the set $\Omega_0^{t,s}$ is isomorphic to the Cartesian product $\Omega^0_{t,u} \times \Omega^0_{u,s}$.

For $t < s$ let $L_g^{t,s}$ denote a restriction of the generalized function L_g on $\Omega^0_{t,s}$. The generalized function $L_g^{t,s}$ is defined on the set of test functions $\mathcal{G}^{t,s} \subset \mathcal{G}$. By definition the function f belongs to $\mathcal{G}^{t,s}$ if $f \in \mathcal{G}$ and for every $\nu \in \Omega^0$

$f(\nu) = f(\nu|_{[t,s)})$ It follows from (5) that on a test function $f \in \mathcal{G}^{t,s}$ a generalized function $L_g^{t,s}$ acts as

$$(L_g^{t,s}, f) = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{[t,s]^k} (g^{\otimes k}, f_k) dm^k. \tag{17}$$

It is clear that for every $u \in (t, s)$ we have

$$L_g^{t,s} = L_g^{t,u} \otimes L_g^{u,s}, \tag{18}$$

where by $L_g^{t,u} \otimes L_g^{u,s}$ we denote the direct product of the generalized functions $L_g^{t,u}$ and $L_g^{u,s}$ (see [8]).

3. Generalized random processes on Ω^0, and corresponding operator semigroups.

In this section we introduce a concept of a generalized random process (below we usually omit the word generalized). To define this random process instead of a probability space we use another object, namely the space $(\Omega^0, \mathcal{G}, L_g)$.

Let $D^0[0,T]$ denote the space of right-continuous step functions on $[0,T]$, with a finite number of jumps. This space is isomorphic to the Cartesian product $\mathbb{R} \times \Omega^0$. Namely, every $x \in \mathbb{R}$ and $\nu \in \Omega^0$ are mapped into the function $f \in D^0[0,T]$, where $f(t) = x + \nu[0,t],\ t \in [0,T]$. On $D^0[0,T]$ we consider the topology that is the image of topology on $\mathbb{R} \times \Omega^0$ under the action of this isomorphism.

We define a generalized process (or simply process) as a measurable (with respect to the Borel $\sigma-$algebra) mapping from Ω^0 to $D^0[0,T]$. Of course, a generalized process is not a random process in usual sense because Ω^0 is not a probability space and instead of a probability measure on Ω^0 we have only a generalized function L_g. For this reason we cannot speak about a probability distribution for a functional of a process. Instead of this we can speak about a generalized distribution that is an image of a generalized function L_g under the action of a functional.

For every $x \in \mathbb{R}$ we define a (generalized) process $\xi_x(t) = \xi_x(t,\nu),\ t \in [0,T],\ x \in \mathbb{R}$, by

$$\xi_x(t) = x + \nu([0,t]). \tag{19}$$

First we study the one-dimensional distribution of the process $\xi_x(t)$. By definition the one dimensional distribution is an image $\xi_x(t)L_g$ of the generalized function L_g under the action of the mapping $\nu \mapsto \xi_x(t,\nu)$. This image is a generalized function on R that acts on a test function φ as

$$(\xi_x(t)L_g, \varphi) = L_g\varphi(\xi_x(t)).$$

Formally the domain (the set of the test functions) of $\xi_x(t)L_g$ is the set of functions φ, such that the function $\nu \mapsto \varphi(\xi_x(t,\nu))$ belongs to $\mathcal{G}$. Let us construct the smaller class of test functions.

For a domain of the generalized function $\xi_x(t)L_g$ we take the set $\mathcal{H}_{fin}$ of functions φ, that are inverse Fourier transforms of signed measures with compact supports and with finite total variations. Namely, each function $\varphi \in \mathcal{H}_{fin}$ is of the form

$$\varphi(x) = \frac{1}{2\pi}\int_{-A}^{A} e^{-ipx}\mu(dp),$$

where μ is a signed measure on $[-A, A]$, such that $|\mu|([-A, A]) < \infty$.

It follows from this definition that for every $\varphi \in \mathcal{H}_{fin}$ there exists $M > 0$ such that

$$\sup_{x\in\mathbb{R}} |\varphi^{(k)}(x)| \leq M^k. \tag{20}$$

for every $k \in \mathbb{N}$,

Now it follows from (16) and (20) that there exists $M_0 > 0$ such that for every $k \in \mathbb{N}$ the following inequality is true

$$\sup_{x\in\mathbb{R}} |(g^{\otimes k}, \varphi(x + x_1 + \cdots + x_k))| \leq M_0^k.$$

Finally, from the latter inequality it follows that for every function $\varphi \in \mathcal{H}_{fin}$ the function $\varphi(\xi_x(t))$ belongs to $\mathcal{G}$.

On the domain $\mathcal{H}_{fin}$ for $t \in [0, T]$ we define a linear operator P^t by

$$P^t\varphi(x) = L_g\varphi(\xi_x(t)) = L_g^{0,t}\varphi(\xi_x(t)) = \sum_{k=0}^{\infty}\frac{t^k}{k!}\big(g^{\otimes k}, \varphi(x + x_1 + \cdots + x_k)\big) =$$

$$\sum_{k=0}^{\infty}\frac{t^k}{k!}\big(g_{x_1}\times\cdots\times g_{x_k}, \varphi(x + x_1 + \cdots + x_k)\big). \tag{21}$$

It follows from (18) that P^t is a semigroup of operators, so that $P^{t+s} = P^tP^s$.

Theorem 1. *The Fourier transform $\widehat{P^t}$ of the operator P^t is a multiplication operator by a function*

$$\widehat{h}_t(p) = \exp(t(g_y, e^{-ipy})),\ \ p \in \mathbb{R}. \tag{22}$$

Proof. For $p \in \mathbb{R},\ \varphi(x) = e^{-ipx}$ we have

$$P^t\varphi(x) = L_g e^{-ip\xi_x(t)} = \sum_{k=0}^{\infty}\frac{t^k}{k!}\big(g^{\otimes k}, e^{-ip(x+x_1+x_2+\cdots+x_k)}\big) =$$

$$e^{-ipx}\exp(t(g_y, e^{-ipy})).$$

□

It follows from (22) that (in the distribution sense) the operator P^t is a convolution operator with the function h_t, where h_t is the inverse Fourier transform of $\widehat{h_t}$.

Further, define a function $u = u(t,x),\ t \geq 0, x \in \mathbb{R}$, by

$$u(t,x) = P^t\varphi(x) = L_g\varphi(\xi_x(t)). \tag{23}$$

Theorem 2. *The function $u(t,x)$ is a solution of the Cauchy problem*

$$\frac{\partial u}{\partial t} = \mathcal{A}_g u, \quad u(0,x) = \varphi(x). \tag{24}$$

Proof. Using (15),(17) we have

$$\frac{\partial u}{\partial t} = \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!}\big(g_{x_1} \times g_{x_2} \times \cdots \times g_{x_k}, \varphi(x + x_1 + x_2 + \cdots + x_k)\big) =$$

$$\left(g_{x_1}, \sum_{k=1}^{\infty} \frac{t^{k-1}}{(k-1)!}\big(g_{x_2} \times g_{x_3} \times \cdots \times g_{x_k}, \varphi(x + x_1 + x_2 + \cdots + x_k)\big)\right).$$

Rename the variable x_1 by y, the variables $x_2, \ldots, x_k$ by $x_1, \ldots, x_{k-1}$ and $k-1$ by k. We get

$$\frac{\partial u}{\partial t} = \left(g_y, \sum_{k=0}^{\infty} \frac{t^k}{k!}\big(g_{x_1} \times g_{x_2} \times \cdots \times g_{x_k}, \varphi(x + y + x_1 + \cdots + x_k)\big)\right) =$$

$$\big(g_y, u(t, x+y)\big) = \mathcal{A}_g u.$$

□

It follows from theorem 2 that the generalized function $\mathcal{P}_{t,x} = \xi_x(t)L_g$ (that is the one-dimensional distribution of the process $\xi_x(t)$) is the fundamental solution of the equation (24). It is easy to compute the Fourier transform $\widehat{\mathcal{P}_{t,x}}$ of this one-dimensional distribution. We have

$$\widehat{\mathcal{P}_{t,x}}(p) = L_g e^{ip\xi_x(t)} = \sum_{k=0}^{\infty} \frac{t^k}{k!}\big(g^{\otimes k}, e^{ip(x+x_1+x_2+\cdots+x_k)}\big) =$$

$$e^{ipx}\exp(t(g_y, e^{ipy})). \tag{25}$$

It is important to note that originally the one-dimensional distribution of the process $\xi_x(t)$ is defined only as a generalized function with domain $\mathcal{H}_{fin}$. But if the Fourier transform (25) is a rapidly decreasing function we can extend $\mathcal{P}_{t,x}$ to a larger class of functions.

Now we consider some examples.

Example 1. Consider the case $g = -\delta^{(1)}$. Then for $\varphi \in \mathcal{H}_{fin}$ we have

$$L_{-\delta^{(1)}}\varphi(\xi_x(t)) = \sum_{k=0}^{\infty} \frac{t^k}{k!}((-\delta^{(1)})^{\otimes k}, \varphi(x + x_1 + \cdots + x_k)) =$$

$$\sum_{k=0}^{\infty} \frac{t^k}{k!}\varphi^{(k)}(x) = \varphi(x+t).$$

The latter formula means that P^t is a shift operator that corresponds to a well-known fact that the fundamental solution $\mathcal{P}_{t,x}$ of the equation $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}$ is the unit mass at the point $x+t$.

Example 2. Consider the case $g = \frac{\delta^{(2)}}{2}$. Then by (22) the operator $\widehat{P^t}$ is a multiplication operator by the function

$$\widehat{h}_t(p) = \exp(t(\frac{\delta_y^{(2)}}{2}, e^{-ipy})) = \exp(-\frac{tp^2}{2}),$$

and then by (25) the Fourier transform $\widehat{\mathcal{P}_{t,x}}$ of the one-dimensional distribution $\mathcal{P}_{t,x}$ (which is the fundamental solution of the equation $\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}$) is

$$\widehat{\mathcal{P}_{t,x}}(p) = e^{ipx - \frac{tp^2}{2}}.$$

Example 3. Consider a generalized function g such that for $\varphi \in \mathcal{H}_{fin}$

$$(g, \varphi) = \text{v.p.} \int_{\mathbb{R}} (\varphi(x) - \varphi(0)) \frac{dx}{x^2}.$$

In this case by (22) we get that the operator $\widehat{P^t}$ is a multiplication operator by the function

$$\widehat{h}_t(p) = \exp\left(t(\text{v.p.} \int_{\mathbb{R}} (e^{-ipy} - 1)\frac{dy}{y^2})\right) = \exp(-\pi t|p|),$$

and the fundamental solution $\mathcal{P}_{t,x}$ of the equation

$$\frac{\partial u}{\partial t}(t,x) = \text{v.p.} \int_{\mathbb{R}} (u(t, x+y) - u(t,x)) \frac{dy}{y^2}$$

is the function

$$\mathcal{P}_{t,x}(y) = \frac{t}{\pi^2 t^2 + (y-x)^2}$$

which is the density of the one-dimensional distribution of the Cauchy process.

The following examples are different from examples 1-3 because the corresponding one-dimensional distributions of processes under consideration are not probability distributions.

Example 4. For $\alpha \in (4m, 4m+2)$ where $m \in \mathbb{N}_0$, we consider the generalized function $g(x) = |x|^{-\alpha-1}$. In this case (see[8])for every $\varphi \in \mathcal{H}_{fin}$ we have

$$(g, \varphi) = \int_{\mathbb{R}} \left(\varphi(x) - \varphi(0) - \frac{\varphi^{(2)}(0)}{2!} x^2 - \cdots - \frac{\varphi^{(4m)}(0)}{(4m)!} x^{4m} \right) \frac{dx}{|x|^{1+\alpha}}. \quad (26)$$

For $\alpha = 4m+1$ the integral (26) is considered in the principal value sense and the case $\alpha = 1$ was considered in the example 3. In this case the operator $\widehat{P^t}$ is the multiplication operator by the function

$$\widehat{h}_t(p) = \exp\left(t \int_{\mathbb{R}} \left(e^{-ipy} - 1 - \frac{(-ip)^2}{2!} y^2 - \cdots - \frac{(-ip)^{4m}}{(4m)!} y^{4m}\right) \frac{dy}{|y|^{1+\alpha}} \right) =$$

$$\exp(-ct|p|^{\alpha}),$$

where

$$c = \frac{\pi}{\sin \frac{\pi\alpha}{2}} \cdot \frac{1}{\Gamma(1+\alpha)} > 0.$$

So the fundamental solution $\mathcal{P}_{t,x}$ (the one-dimensional distribution of our process) of the equation

$$\frac{\partial u}{\partial t}(t,x) =$$

$$\int_{\mathbb{R}} \left(u(t, x+y) - u(t,x) - \frac{1}{2!} \frac{\partial^2 u}{\partial x^2}(t,x) y^2 - \cdots - \frac{1}{(4m)!} \frac{\partial^{4m} u}{\partial x^{4m}}(t,x) y^{4m} \right) \frac{dy}{|y|^{1+\alpha}}$$

is the function

$$\mathcal{P}_{t,x}(y) = h_t(y-x),$$

where h_t is the inverse Fourier transform of the function $\widehat{h}_t$. This solution is a probability distribution only if $m = 0, \alpha \in (0,2)$.

Example 5. Now we consider the same generalized function $g(x) = |x|^{-\alpha-1}$, as in the previous example but for $\alpha \in (4m-2, 4m)$ and $m \in \mathbb{N}$. In this case

$$(g, \varphi) = \int_{\mathbb{R}} \left(\varphi(x) - \varphi(0) - \frac{\varphi^{(2)}(0)}{2!} x^2 - \cdots - \frac{\varphi^{(4m-2)}(0)}{(4m-2)!} x^{4m-2} \right) \frac{dx}{|x|^{1+\alpha}}, \quad (27)$$

the $\widehat{P^t}$ is a multiplication operator by the function $\exp(ct|p|^{\alpha})$, but in this case $c > 0$, that means that the fundamental solution of the corresponding equation is only a generalized function with a domain $\mathcal{H}_{fin}$ and it cannot be extend to a larger class of test function.

Note that if instead of (27) we consider another generalized function (here A is an arbitrary positive number)

$$(g, \varphi) = \int_{\mathbb{R}} \left(\varphi(x) - \varphi(0) - \frac{\varphi^{(2)}(0)}{2!} x^2 - \ldots \right.$$

$$-\frac{\varphi^{(4m-2)}(0)}{(4m-2)!}x^{4m-2}-\frac{\varphi^{(4m)}(0)}{(4m)!}x^{4m}\mathbf{1}_{[-A,A]}(x)\Big)\frac{dx}{|x|^{1+\alpha}}, \tag{28}$$

then this problem disappears. Namely, the fundamental solution $\mathcal{P}_{t,x}$ of the equation

$$\frac{\partial u}{\partial t}(t,x)=\int_{\mathbb{R}}\Big(u(t,x+y)-u(t,x)-\frac{1}{2!}\frac{\partial^2 u}{\partial x^2}(t,x)y^2-\cdots-$$

$$\frac{1}{(4m-2)!}\frac{\partial^{4m-2}u}{\partial x^{4m-2}}(t,x)y^{4m}\Big)\frac{dy}{|y|^{1+\alpha}}-\varepsilon\frac{\partial^{4m}u}{\partial x^{4m}}(t,x)$$

(here $\varepsilon>0$ is a constant which depends on A) is the function

$$\mathcal{P}_{t,x}(y)=h_t(y-x),$$

where h_t is the inverse Fourier transform of the function

$$\widehat{h}_t(y)=\exp(t(c|p|^\alpha-\varepsilon_1 p^{4m})).$$

Example 6. Consider the generalized function

$$g=(-1)^{m+1}\frac{\delta^{(2m)}}{(2m)!}.$$

This generalized function corresponds to the pseudo-process of even order discussed in [6]. In this case $\widehat{P^t}$ is the multiplication operator by the function

$$\widehat{h}_t(p)=\exp(t((-1)^{m+1}\frac{\delta^{(2m)}}{(2m)!},e^{-ipy}))=\exp(-\frac{tp^{2m}}{(2m)!}),$$

so that the fundamental solution $\mathcal{P}_{t,x}$ of the equation

$$\frac{\partial u}{\partial t}=\frac{(-1)^{m+1}}{(2m)!}\frac{\partial^{2m}u}{\partial x^{2m}}$$

is the function

$$\mathcal{P}_{t,x}(y)=h_t(y-x),$$

where h_t is the inverse Fourier transform of the function $\widehat{h}_t(p)$.

Similarly, to the case of the pseudo-process of odd order in our construction corresponds $g=\pm\frac{\delta^{(2m+1)}}{(2m+1)!}$ and

$$\widehat{h}_t(p)=\exp(t(\pm\frac{\delta^{(2m+1)}}{(2m+1)!},e^{-ipy}))=\exp(\pm(-1)^m\frac{itp^{2m+1}}{(2m+1)!}).$$

Example 7. For every $x\in\mathbb{R}$ we define on Ω^0 another process $\xi_x^c(t)=\xi_x^c(t,\nu)$, by

$$\xi_x^c(t)=x+c\nu([0,t]), \tag{29}$$

where $c \in \mathbb{C}$ is a complex constant. Now we consider the one-dimensional distribution of the process $\xi_x^c(t)$. We recall that by definition one-dimensional distribution is a generalized function $\xi_x^c(t)L_g$ that acts on a test function $\varphi \in \mathcal{H}_{fin}$ as

$$(\xi_x^c(t)L_g, \varphi) = L_g\varphi(\xi_x^c(t)).$$

As above, for $\varphi \in \mathcal{H}_{fin}$ we define a function $u = u(t,x),\ t \geq 0, x \in \mathbb{R}$, by

$$u(t,x) = L_g\varphi(\xi_x^c(t)). \tag{30}$$

It is important to note that in (30) we actually substitute $\xi_x^c(t)$ in the analytic continuation of φ.

In the same way we can prove that the function u is a solution of the Cauchy problem

$$\frac{\partial u}{\partial t} = \mathcal{A}_g^c u, \quad u(0,x) = \varphi(x),$$

where the linear operator $\mathcal{A}_g^c$ is defined by

$$\mathcal{A}_g^c f(x) = \big(g_y, f(x+cy)\big). \tag{31}$$

In the special case $g = \frac{\delta^{(2)}}{2}$, we have $\mathcal{A}_g^c f(x) = \frac{c^2}{2} f^{(2)}(x)$. So, if we put $c = e^{\frac{-i\pi}{4}}$ in (29) we get that one-dimensional distribution of the process

$$\xi_x(t) = x + e^{\frac{-i\pi}{4}}\nu[0,t] \tag{32}$$

give us the fundamental solution of the Schrödinger equation

$$i\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}.$$

Further, in the case $g = \frac{\delta^{(4m)}}{(4m)!}$ (note, that in the example 6 we consider another case with $g = -\frac{\delta^{(4m)}}{(4m)!}$) we get $\mathcal{A}_g^c f(x) = \frac{c^{4m}}{(4m)!} f^{(4m)}(x)$. If we put $c = e^{\frac{i\pi}{4m}}$ in (29) we get that the one-dimensional distribution of the process

$$\xi_x(t) = x + e^{\frac{i\pi}{4m}}\nu[0,t] \tag{33}$$

gives us the fundamental solution of the equation

$$\frac{\partial u}{\partial t} = \frac{-1}{(4m)!}\frac{\partial^{4m} u}{\partial x^{4m}}. \tag{34}$$

4. The distributions of functionals. The stochastic integrals.

As we have shown one-dimensional distributions of generalized processes with independent increments are the fundamental solutions of the corresponding evolution equations. But it is possible to consider some other functionals of the trajectories of the (generalized) processes.

First we consider the variations of trajectories. Let $g = \frac{\delta^{(2)}}{2}$, as we have seen this case corresponds to the Wiener process. Consider the quadratic variation v_2 of a process ξ_0. We suppose here that the process starts at point 0 as the quadratic variation does not depend on a shift. It is clear that for $\nu = \sum_{k=1}^{n} x_k \delta_{t_k}$ and $\xi_0(t) = \nu([0,t])$ we have

$$v_2(\xi_0) = \sum_{k=1}^{n} x_k^2.$$

Now we consider the distribution $\mathcal{P}_{v_2(\xi_0)} = v_2(\xi_0) L_{\frac{\delta^{(2)}}{2}}$. By definition $v_2(\xi_0) L_{\frac{\delta^{(2)}}{2}}$ is a generalized function that acts on a test function $\varphi \in \mathcal{H}_{fin}$ as

$$(\mathcal{P}_{v_2(\xi_0)}, \varphi) = L_{\frac{\delta^{(2)}}{2}} \varphi(v_2(\xi_0)) =$$

$$\sum_{k=0}^{\infty} \frac{T^k}{k! 2^k} \big((\delta^{(2)})^{\otimes k}, \varphi(x_1^2 + \cdots + x_k^2) \big) = \sum_{k=0}^{\infty} \frac{T^k}{k!} \varphi^{(k)}(0) = \varphi(T). \tag{35}$$

The latter equality means that the distribution $\mathcal{P}_{v_2(\xi_0)}$ is a unit mass at point T, that expresses a well-known fact that the quadratic variation of the Wiener process is not random and equal to T with probability 1.

This statement can be easily extended to the case (example 6)

$$g = (-1)^{m+1} \frac{\delta^{(2m)}}{(2m)!}.$$

In this case we consider v_{2m}, the variation of order $2m$ instead of the quadratic variation v_2. As above for $\nu = \sum_{k=1}^{n} x_k \delta_{t_k}$ and $\xi_0(t) = \nu([0,t])$ we have

$$v_{2m}(\xi_0) = \sum_{k=1}^{n} x_k^{2m}.$$

The distribution $\mathcal{P}_{v_{2m}(\xi_0)} = v_{2m}(\xi_0) L_{\frac{(-1)^{m+1}\delta^{(2m)}}{(2m)!}}$ acts on a test function $\varphi \in \mathcal{H}_{fin}$ as

$$(\mathcal{P}_{v_{2m}(\xi_0)}, \varphi) = L_{\frac{(-1)^{m+1}\delta^{(2m)}}{(2m)!}} \varphi(v_{2m}(\xi_0)) =$$

$$\sum_{k=0}^{\infty} \frac{T^k((-1)^{m+1})^k}{k!((2m)!)^k} \big((\delta^{(2m)})^{\otimes k}, \varphi(x_1^{2m} + \cdots + x_k^{2m})\big) =$$

$$\sum_{k=0}^{\infty} \frac{T^k((-1)^{m+1})^k}{k!} \varphi^{(k)}(0) = \varphi((-1)^{m+1}T). \tag{36}$$

Thus we prove that the order $2m$ variation of the process (with respect to the generalized function $L_{\frac{(-1)^{m+1}\delta^{(2m)}}{(2m)!}}$) is constant but for even m this constant is negative. Below we explain the reason for this strange result. Namely we show that a generalized process has a probabilistic sense not for all generalized functions g, but only for nonnegative (in some sense) g. For a generalized function g of the form $(g, \varphi) = a\varphi^{(k)}(0)$, where a is constant, this nonnegativity means that a is nonnegative. So for even $m = 2n$ only a complex-valued process (33) (with respect to a generalized function $L_{\frac{\delta^{(4n)}}{(4n)!}}$) from the example 7 has a probabilistic meaning. It is absolutely clear that the variation of the order $4n$ for this process is negative.

The next object we define is a stochastic integral with respect to a generalized process $\xi(t)$. Let $f : [0, T] \to \mathbb{R}$ be a Borel function and $\xi(t) = \nu[0, t]$. For every trajectory $\xi(\cdot)$ we define a stochastic integral $\int_0^T f(t)d\xi(t)$ as a usual Stieltjes integral with respect to the jump function $\xi(\cdot)$ (we recall that trajectories of our process have only a finite number of jumps). Namely for $\nu = \sum_{j=1}^k x_j \delta_{t_j} \in \Omega^0$ we put

$$I_1(f) = \int_0^T f(t)d\xi(t) = \int_0^T f(t)d\nu(t) = \sum_{1 \le j \le k} f(t_j)x_j,$$

(by definition this sum is equal to 0 if $k = 0$).

To construct a distributions $\mathcal{P}_{I_1(f)} = I_1(f)L_g$ corresponding a generalized function g we calculate the Fourier transform $\widehat{\mathcal{P}_{I_1(f)}}(p)$ of a distribution $\mathcal{P}_{I_1(f)}$.

We have

$$\widehat{\mathcal{P}_{I_1(f)}}(p) = L_g e^{ip\int_0^T f(t)d\xi(t)} = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{[0,T]^k} \big(g^{\otimes k}, e^{ip\sum_{j=1}^k f(t_j)x_j}\big) dm^k =$$

$$\exp\left(\int_0^T (g_x, e^{ipf(t)x}) dt \right). \tag{37}$$

Let us consider some examples.

Example 1. For $g = -\delta^{(1)}$ (so that $(g, \varphi) = \varphi'(0)$) using (37) we have

$$\widehat{\mathcal{P}_{I_1(f)}}(p) = \exp\left(ip \int_0^T f(t)dt \right). \tag{38}$$

It follows from (38) that the distribution of the stochastic integral is a $\delta-$measure at the point $\int_0^T f(t)dt$. This means that the stochastic integral is nonrandom and coincides with the Lebesgue integral.

Example 2.

For $g = \frac{\delta^{(2)}}{2}$ we have

$$\widehat{\mathcal{P}_{I_1(f)}}(p) = \exp\big(-\frac{p^2}{2}\int_0^T f^2(t)dt\big),$$

hence the corresponding distribution is a normal one with parameters $(0, \int_0^T f^2(t)dt)$.

Example 3. For $g = \frac{(-1)^{m+1}\delta^{(2m)}}{(2m)!}$ we have

$$\widehat{\mathcal{P}_{I_1(f)}}(p) = \exp\big(-\frac{p^{2m}}{(2m)!}\int_0^T f^{2m}(t)dt\big).$$

Note that for $m > 1$ the corresponding distribution is not a probability distribution.

Example 4.

For $g(x) = |x|^{-\alpha-1}$, and $\alpha \in (4m, 4m+2)$ where $m \in \mathbb{N}_0$ (example 4 of the section 3), it is not difficult to prove that

$$\widehat{\mathcal{P}_{I_1(f)}}(p) = \exp\big(-c|p|^\alpha \int_0^T |f(t)|^\alpha dt\big)$$

where c is a positive constant.

The corresponding distribution is a symmetric stable distribution if $m = 0$ and it is a signed measure if $m > 0$.

Similarly we can define multiple stochastic integrals. Namely, let $f : [0,T]^n \to \mathbb{R}$ be a Borel function of n arguments. We define a stochastic integral $I_n(f)$ of multiplicity n in the following way.

For $\nu = \sum_{j=1}^k x_j\delta_{t_j} \in \Omega^0$ we set

$$I_n(f) = \int_{[0,T]^n} f(t_1,\dots,t_n)d\xi(t_1)\dots d\xi(t_n) =$$

$$\sum_{j_1\neq j_2\dots\neq j_n} f(t_{j_1}, t_{j_2}\dots t_{j_n})x_{j_1}x_{j_2}\dots x_{j_n}$$

(this sum is equal to 0 if $k < n$).

We show that for $g = -\delta^{(1)}$ this multiple stochastic integral is nonrandom and coincides with the multiple Lebesgue integral.

Theorem 3. *If $g = -\delta^{(1)}$ and $f \in L_1([0,T]^n)$ then for every $n \in \mathbb{N}$ the distribution $\mathcal{P}_{I_n(f)} = I_n(f)L_{-\delta^{(1)}}$ of the multiple stochastic integral is a $\delta-$measure at the point $\int_{[0,T]^n} f(t_1,\dots,t_n)dt_1\dots dt_n$.*

Proof. For simplicity we consider only the case $n = 2$.

To prove the statement it is sufficient to check the following equality

$$L_{-\delta^{(1)}} e^{ipI_2(f)} = \exp\big(ip\int_{[0,T]^2} f(t_1,t_2)dt_1dt_2\big) \tag{39}$$

for every $p \in \mathbb{R}$. We have

$$L_{-\delta^{(1)}} e^{ipI_2(f)} = \sum_{d=0}^{\infty} \frac{(ip)^d L_{-\delta^{(1)}}(I_2(f))^d}{d!}$$

Further, for every d we have

$$L_{-\delta^{(1)}}(I_2(f))^d = \sum_{k=2}^{\infty} \frac{1}{k!}\int_{[0,T]^k} dm^k \Bigg((-\delta^{(1)})^{\otimes k}, \big(\sum_{\substack{j,l=1\\ j\neq l}}^{k} f(t_j,t_l)x_jx_l\big)^d\Bigg).$$

It is easy to show that all terms in the sum on the right hand side, except the term with $k = 2d$, are equal to 0 and the term with $k = 2d$ is equal to $\big(\int_{[0,T]^2} f(t_1,t_2)dt_1dt_2\big)^d$. This proves (39). □

Further, we can show that when $g = \frac{\delta^{(2)}}{2}$ a multiple stochastic integral coincides with a multiple stochastic integral with respect to the Wiener process. We omit the proof since it is completely similar to the proof of theorem 3.

Theorem 4. *If $g = \frac{\delta^{(2)}}{2}$ and $f \in L_2([0,T]^n)$ then for every $n \in \mathbb{N}$ the distribution $\mathcal{P}_{I_n(f)} = I_n(f)L_{\frac{\delta^{(2)}}{2}}$ of the multiple stochastic integral coincides with the distribution of the Wiener multiple stochastic integral*

$$\int_{[0,T]^n} f(t_1,\ldots,t_n)dw(t_1)\ldots dw(t_n)$$

and moreover for $n \neq m$ and any functions $f_1 \in L_2([0,T]^n)$, $f_2 \in L_2([0,T]^m)$, we have

$$L_{\frac{\delta^{(2)}}{2}}(I_n(f_1)I_m(f_2)) = 0.$$

5. The probability space $(\Omega, \mathcal{F}, P_g)$.

In this section we additionally suppose that a generalized function g is either a regularization of a nonnegative function (previously we did not assume nonnegativity) or $g = a(-1)^r\delta^{(r)}$, for some $a > 0$.

Let us first consider the case where g is a nonnegative function satisfying the condition $\int_{\mathbb{R}} \min(|x|^r, 1)g(x)dx < \infty$ for some $r \in \mathbb{N}$ (we use the same letter g for the function itself and for the corresponding generalized function).

We put

$$n_0 = \max\{n : \int_{|x|>1} |x|^n g(x)dx < \infty\}. \tag{40}$$

Let $\mathcal{X} = \mathcal{X}(G)$, be the space of configurations on $G = [0,T] \times (\mathbb{R} \setminus \{0\})$. We equip $\mathcal{X}$ with the vague topology (see [12]) and denote by $\mathcal{B}(\mathcal{X})$ the Borel $\sigma-$algebra. On the measurable space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ we consider the Poisson measure $\mathbb{P}_g$ with intensity measure $\Pi(dt, dx) = dtg(x)dx$.

Further, for every $\varepsilon > 0$ we denote a restriction of the measure $\mathbb{P}_g$ to the space $\mathcal{X}(G_\varepsilon)$ of configurations on $G_\varepsilon = [0,T] \times (\mathbb{R} \setminus [-\varepsilon, \varepsilon])$ by $\mathbb{P}_{g,\varepsilon}$. It is clear that for every ε the measure $\mathbb{P}_{g,\varepsilon}$ is a Poisson measure with an intensity measure $\Pi |_{G_\varepsilon}$. As before we denote by m the Lebesgue measure on $[0,T]$.

For a probability space we take the space $\Omega = \Omega([0,T])$ of all discrete signed measures on $[0,T]$. Each element of this space can be represented in the form $\sum_j x_j \delta_{t_j}$, where as above we denote the unit mass at point t_j by δ_{t_j} Note that $\Omega^0 \subset \Omega$.

Let Θ denote the mapping $\mathcal{X} \to \Omega$ that maps a configuration $\bigcup_i (t_i, x_i) \in \mathcal{X}$ to a signed measure $\sum_i x_i \delta_{t_i} \in \Omega$.

Using the mapping Θ we equip Ω with a topology and a Borel $\sigma-$algebra $\mathcal{F}$. In this way on Ω we construct probability measures $P_g = \mathbb{P}_g \Theta^{-1}$ and $P_{g,\varepsilon} = \mathbb{P}_{g,\varepsilon} \Theta^{-1}$. It is natural to call these measures Poisson measures. Note that for every $\varepsilon > 0$ $P_{g,\varepsilon}(\Omega^0) = 1$.

Let us study the connection between a generalized function L_g on Ω^0 and a measure P_g on Ω.

Let v be a $C^\infty-$smooth function on $\mathbb{R}$ and d denote an operator

$$(dv)(x) = v(0) = \text{Const}.$$

For $k = 1, 2, 3, \dots$ let $d^{(k)}$ denote a linear operator that acts as

$$(d^{(k)}v)(x) = \frac{x^k}{k!} v^{(k)}(0) \tag{41}$$

for $k \le n_0$, and

$$(d^{(k)}v)(x) = \frac{x^k}{k!} \mathbf{1}_{[0,1]}(|x|) v^{(k)}(0) \tag{42}$$

for $k > n_0$. It is important to note that due to the difference between (41) and (42) $d^{(k)}$ depends on g.

Further, we define a linear operator Δ by

$$\Delta v(x) = v(x) - v(0) = v(x) - dv(x), \tag{43}$$

and define a sequence of operators (which also depend on g) $\Delta(k), k \in \mathbb{N}$ by

$$(\Delta(k)v)(x) = v(x) - dv(x) - d^{(1)}v(x) - \cdots - d^{(k)}v(x).$$

Note that $\Delta + d$ is the identity operator and for every $k = 1, 2, \dots$ we have

$$\Delta(k) = \Delta(k+1) + d^{(k+1)}. \tag{44}$$

For $r > 1$ and a test function φ we have

$$(g, \varphi) = \int_{\mathbb{R}} \Delta(r-1)\varphi(x)g(x)dx = \lim_{\varepsilon \to 0} \int_{\mathbb{R}_\varepsilon} \Delta(r-1)\varphi(x)g(x)dx, \tag{45}$$

where $\mathbb{R}_\varepsilon = \mathbb{R} \setminus [-\varepsilon, \varepsilon]$, and for $r = 1$ and a test function φ we have

$$(g, \varphi) = \int_{\mathbb{R}} \Delta\varphi(x)g(x)dx = \lim_{\varepsilon \to 0} \int_{\mathbb{R}_\varepsilon} \Delta\varphi(x)g(x)dx.$$

We also set

$$\tau_l(\varepsilon) = \int_{\mathbb{R}_\varepsilon} x^l g(x)dx, \tag{46}$$

for $\varepsilon > 0$ and $l \leq n_0$ and

$$\tau_l(\varepsilon) = \int_{\varepsilon \leq |x| \leq 1} x^l g(x)dx \tag{47}$$

for $l > n_0$.

Note that for an even function g and odd l all the functions $\tau_l(\varepsilon)$ are identically equal to 0.

First we consider the case $r = 1$.

Theorem 5. *Suppose that $r = 1$. Then for every $f \in \mathcal{G}$*

$$\lim_{\varepsilon \to 0} \int_\Omega f dP_{g,\varepsilon} = L_g f,$$

that means, that for $r = 1$ the generalized function L_g is a limit (in generalized functions sense, that is on each test function) of the probability measures $P_{g,\varepsilon}$.

Proof. For $\varepsilon > 0$, $\quad f \in \mathcal{G}$ we have

$$\int_\Omega f dP_{g,\varepsilon} = e^{-\Pi(G_\varepsilon)} \sum_{k=0}^{\infty} \frac{1}{k!} \int_{G_\varepsilon^k} f_k d\Pi^k. \tag{48}$$

Denote the action of the operators Δ, d with respect to the variable x_i by Δ_i, d_i.

For every finite set I of natural numbers let $|I|$ denote the cardinality of the set I. We put

$$d_I = \prod_{i \in I} d_i, \quad \Delta_I = \prod_{i \in I} \Delta_i. \tag{49}$$

If $I=\{1,\dots,k\}$ then for the corresponding operator Δ_I we use a notation $\Delta^{\otimes k}$ that is

$$\Delta^{\otimes k}=\Delta_1\Delta_2\dots\Delta_k.$$

We use as well corresponding notations for operators
$\Delta(m),\ d^{(m)},\ m\in\mathbb{N}$, that is

$$d_I^{(m)}=\prod_{i\in I}d_i^{(m)},\quad \Delta_I(m)=\prod_{i\in I}\Delta_i(m)$$

and

$$\Delta^{\otimes k}(m)=\Delta_1(m)\Delta_2(m)\dots\Delta_k(m).$$

For every fixed k, let CI denote the set $\{1,\dots,k\}\setminus I$. We note that for any k the function $d_{CI}f_k$ depends only on $x_i,\ i\in I$.

Using the identity

$$1=\prod_{i=1}^{k}(d_i+\Delta_i)=\sum_{I\subset\{1,\dots,k\}}\Delta_I d_{CI},$$

and (48), we get

$$\int_\Omega f dP_{g,\varepsilon}=e^{-\Pi(G_\varepsilon)}\sum_{k=0}^{\infty}\frac{1}{k!}\int_{G_\varepsilon^k}\prod_{i=1}^{k}(\Delta_i+d_i)f_k d\Pi^k=$$

$$e^{-\Pi(G_\varepsilon)}\sum_{k=0}^{\infty}\frac{1}{k!}\int_{G_\varepsilon^k}\sum_{I\subset\{1,\dots,k\}}\Delta_I d_{CI}f_k d\Pi^k=$$

$$e^{-\Pi(G_\varepsilon)}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{I\subset\{1,\dots,k\}}\Pi(G_\varepsilon)^{|CI|}\int_{G_\varepsilon^{|I|}}\Delta_I d_{CI}f_k\prod_{i\in I}\Pi(dx_i)=$$

$$e^{-\Pi(G_\varepsilon)}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{j=0}^{k}\Pi(G_\varepsilon)^{k-j}\binom{k}{j}\int_{G_\varepsilon^j}\Delta^{\otimes j}f_j d\Pi^j=$$

$$e^{-\Pi(G_\varepsilon)}\sum_{j=0}^{\infty}\frac{1}{j!}\int_{G_\varepsilon^j}\Delta^{\otimes j}f_j d\Pi^j\sum_{k=j}^{\infty}\frac{\Pi(G_\varepsilon)^{k-j}}{(k-j)!}=\sum_{k=0}^{\infty}\frac{1}{k!}\int_{G_\varepsilon^k}\Delta^{\otimes k}f_k d\Pi^k. \tag{50}$$

Thus, we have

$$\int_\Omega f dP_{g,\varepsilon}=\sum_{k=0}^{\infty}\frac{1}{k!}\int_{G_\varepsilon^k}\Delta^{\otimes k}f_k d\Pi^k\underset{\varepsilon\to 0}{\longrightarrow}\sum_{k=0}^{\infty}\frac{1}{k!}\int_{G^k}\Delta^{\otimes k}f_k d\Pi^k=L_g f.$$

□

Let us introduce some additional notations. Namely, we define a sequence $\{\mathcal{D}_j\}_{j=1}^{\infty}$ of differential operators For $f \in \mathcal{G}$ we define as follows $\mathcal{D}_1 f : \Omega^0 \to \mathbb{R}$, such that for every $k \in \mathbb{N}_0$

$$(\mathcal{D}_1 f)_k((t_1, x_1), \dots, (t_k, x_k)) =$$

$$\int_0^T dt_{k+1} \frac{\partial}{\partial x_{k+1}} f_{k+1}((t_1, x_1), \dots, (t_k, x_k), (t_{k+1}, x_{k+1})) \mid_{x_{k+1}=0} .$$

By analogy we define the differential operators $\mathcal{D}_2, \mathcal{D}_3, \dots$. Namely, for $m = 2, 3, \dots$ we set

$$(\mathcal{D}_m f)_k((t_1, x_1), \dots, (t_k, x_k)) =$$

$$\int_0^T dt_{k+1} \frac{\partial^m}{\partial x_{k+1}^m} f_{k+1}((t_1, x_1), \dots, (t_k, x_k), (t_{k+1}, x_{k+1})) \mid_{x_{k+1}=0} .$$

For $t \in \mathbb{R}$ we also denote the corresponding operator exponents by

$$Q_1^t = e^{t\mathcal{D}_1},\ Q_2^t = e^{t\frac{\mathcal{D}_2}{2!}}, \dots,\ Q_m^t = e^{t\frac{\mathcal{D}_m}{m!}}, \dots .$$

Consider properties of the operator Q_1^t. It is easy to show that,

$$Q_1^t(\nu[0,T]) = \nu[0,T] + tT.$$

Moreover it is not hard to prove (see [2]) that properties of the operator Q_1^t are similar to the properties of a shift operator. For example for every $\varphi \in \mathcal{H}_{fin}$ we have $Q_1^t(\varphi(f)) = \varphi(Q_1^t(f))$. In particular we have $Q_1^t(\varphi(\nu[0,T])) = \varphi(\nu[0,T] + tT)$. For this reason we call Q_1^t generalized shift operators.

Now we consider the case $r = 2$, so that $\int_{\mathbb{R}} \min(x,1) g(x) dx = \infty$ and $\int_{\mathbb{R}} \min(x^2,1) g(x) dx < \infty$.

Using (44),(50) we have

$$\int_\Omega f dP_{g,\varepsilon} = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{G_\varepsilon^k} \Delta^{\otimes k} f_k d\Pi^k = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{G_\varepsilon^k} \prod_{i=1}^{k} (\Delta_i(1) + d_i^{(1)}) f_k d\Pi^k =$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} \int_{G_\varepsilon^k} \sum_{I \subset \{1,\dots,k\}} \Delta_I(1) d_{CI}^{(1)} f_k d\Pi^k =$$

$$\sum_{k=0}^{\infty} \frac{1}{k!} \int_{G_\varepsilon^k} \sum_{j=0}^{k} \binom{k}{j} \Delta^{\otimes j}(1) (d^{(1)})^{\otimes(k-j)} f_k d\Pi^k =$$

(in the latter formula we suppose that the operator $\Delta^{\otimes j}(1)$ acts with respect to the variables $x_1, x_2, \dots, x_j$, and the operator $(d^{(1)})^{\otimes(k-j)}$ acts with respect to the other $k-j$ variables)

$$\sum_{k=0}^{\infty} \int_{G_\varepsilon^k} \sum_{j=0}^{k} \frac{1}{j!(k-j)!} \Delta^{\otimes j}(1) (d^{(1)})^{\otimes(k-j)} f_k d\Pi^k =$$

$$\sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{1}{j!(k-j)!}\int_{G_\varepsilon^k}\Delta^{\otimes j}(1)\mathcal{D}_1^{k-j}f_k(\tau_1(\varepsilon))^{k-j}d\Pi^j =$$
$$\sum_{j=0}^{\infty}\frac{1}{j!}\int_{G_\varepsilon^j}\Delta^{\otimes j}(1)\big(\sum_{k=0}^{\infty}\frac{1}{k!}\mathcal{D}_1^k f_{j+k}(\tau_1(\varepsilon))^k\big)d\Pi^j =$$
$$\sum_{j=0}^{\infty}\frac{1}{j!}\int_{G_\varepsilon^j}\Delta^{\otimes j}(1)\big(Q_1^{\tau_1(\varepsilon)}f\big)_j d\Pi^j.$$

Rename $Q_1^{\tau_1(\varepsilon)}f$ as f. We get

$$\int Q_1^{-\tau_1(\varepsilon)}f dP_{g,\varepsilon}=\sum_{k=0}^{\infty}\frac{1}{k!}\int_{G_\varepsilon^k}\Delta^{\otimes k}(1)f_k d\Pi^k. \tag{51}$$

Denote the operator $Q_1^{-\tau_1(\varepsilon)}$ by $\mathcal{A}_\varepsilon$, and its conjugate operator (which acts on measures), by $\mathcal{A}_\varepsilon^*$, namely for $f\in\mathcal{G}$ we set

$$(\mathcal{A}_\varepsilon^* P_{g,\varepsilon},f)=\int\mathcal{A}_\varepsilon f dP_{g,\varepsilon}=\sum_{k=0}^{\infty}\frac{1}{k!}\int_{G_\varepsilon^k}\Delta^{\otimes k}(1)f_k d\Pi^k. \tag{52}$$

Then as $\varepsilon\to 0$ the right hand side of (52) converges to a limit

$$\sum_{k=0}^{\infty}\frac{1}{k!}\int_{G^k}\Delta^{\otimes k}(1)f_k d\Pi^k = L_g f.$$

Thus, we have proved the following statement.

Theorem 6. *Suppose that $r=2$. Then for every $f\in\mathcal{G}$*

$$\lim_{\varepsilon\to 0}\int_\Omega\mathcal{A}_\varepsilon f dP_{g,\varepsilon}=\lim_{\varepsilon\to 0}(\mathcal{A}_\varepsilon^* P_{g,\varepsilon},f)=L_g f,$$

where $\mathcal{A}_\varepsilon$ is a generalized shift operator. This means, that for $r=2$ a generalized function L_g is a limit of generalized functions $\mathcal{A}_\varepsilon^* P_{g,\varepsilon}$.

Now consider the case $r>2$. By the same arguments we can prove that

$$\int Q_{r-1}^{-\tau_{r-1}(\varepsilon)}Q_{r-2}^{-\tau_{r-2}(\varepsilon)}\dots Q_1^{-\tau_1(\varepsilon)}f dP_{g,\varepsilon}=\sum_{k=0}^{\infty}\frac{1}{k!}\int_{G_\varepsilon^k}\Delta^{\otimes k}(r-1)f_k d\Pi^k. \tag{53}$$

Denote an operator $Q_{r-1}^{-\tau_{r-1}(\varepsilon)}Q_{r-2}^{-\tau_{r-2}(\varepsilon)}\dots Q_1^{-\tau_1(\varepsilon)}$ by $\mathcal{A}_\varepsilon$ and its conjugate operator by $\mathcal{A}_\varepsilon^*$. We have

$$(\mathcal{A}_\varepsilon^* P_{g,\varepsilon},f)=\int\mathcal{A}_\varepsilon f dP_{g,\varepsilon}=$$

$$\sum_{k=0}^{\infty}\frac{1}{k!}\int_{G_\varepsilon^k}\Delta^{\otimes k}(r-1)f_k d\Pi^k \underset{\varepsilon\to 0}{\longrightarrow}\sum_{k=0}^{\infty}\frac{1}{k!}\int_{G^k}\Delta^{\otimes k}(r-1)f_k d\Pi^k = L_g f.$$

Thus, we have proved that $\mathcal{A}_\varepsilon^* P_{g,\varepsilon}$ converges to a generalized function L_g as $\varepsilon \to 0$, and we have the following statement.

Theorem 7. *Suppose that $r > 2$. Then for every $f \in \mathcal{G}$ we have*

$$\lim_{\varepsilon\to 0}\int_\Omega \mathcal{A}_\varepsilon f dP_{g,\varepsilon} = \lim_{\varepsilon\to 0}(\mathcal{A}_\varepsilon^* P_{g,\varepsilon}, f) = L_g f,$$

where operators $\mathcal{A}_\varepsilon$ are defined by (53). That means, that the generalized function L_g is a limit of generalized functions $\mathcal{A}_\varepsilon^* P_{g,\varepsilon}$.

The following theorem shows the actions of the operator $\mathcal{A}_\varepsilon$ on the functions that are of the form $\varphi(\nu[0,T])$.

Theorem 8. *For every $\varphi \in \mathcal{H}_{fin}$ we have*

$$\mathcal{A}_\varepsilon(\varphi(\nu[0,T])) = (b_\varepsilon * \varphi)(\nu[0,T]))$$

where the Fourier transform $\widehat{b_\varepsilon}(p)$ of the function b_ε is

$$\widehat{b_\varepsilon}(p) = \exp\biggl(-T\biggl[\frac{(-ip)^{r-1}\tau_{r-1}(\varepsilon)}{(r-1)!}+\cdots+\frac{(-ip)^2\tau_2(\varepsilon)}{2!}+\frac{-ip\tau_1(\varepsilon)}{1!}\biggr]\biggr).$$

The proof can be found in [16].

Now we consider the case where the generalized function g acts on a test function φ as

$$(g,\varphi) = a\varphi^{(r)}(0),$$

where $a > 0$ and $r \in \mathbb{N}$. Without loss of generality we can assume that $a = \frac{1}{r!}$. In this case $\mathcal{A} = \frac{1}{r!}\frac{d^r}{dx^r}$.

We remark that in the previous case when g was a regularization of a nonnegative function our arguments were based on the formula (45). Now instead of (45) we use another formula. Namely, it can easily be proved that for $\varphi \in \mathcal{H}_{fin}$ we have

$$(g,\varphi) = \frac{1}{r!}\varphi^{(r)}(0) = \lim_{\varepsilon\to 0}\int_\varepsilon^{e\varepsilon}\Delta(r-1)\varphi(x)\frac{dx}{x^{r+1}} \tag{54}$$

(where e stands for the base of the natural logarithm).

As above we take the space $\Omega = \Omega([0,T])$ as a probability space and on $(\Omega, \mathcal{F})$ we consider the Poisson measure P_g with intensity measure $\Pi(dt,dx) = dt\frac{dx}{x^{1+r}}\mathbf{1}_{(0,+\infty)}(x)$. By $P_{g,\varepsilon}$ we denote the Poisson measure with intensity measure $dt\frac{dx}{x^{1+r}}\mathbf{1}_{(\varepsilon,e\varepsilon)}(x)$. It is obvious that, for every $\varepsilon > 0$, $P_{g,\varepsilon}(\Omega^0) = 1$.

We also set

$$G_\varepsilon^0 = [0,T] \times [\varepsilon, e\varepsilon].$$

Now we study the connection between the generalized function L_g on Ω^0 and the measure P_g on Ω.

For $l < r$ we set

$$\tau_l^0(\varepsilon) = \int_\varepsilon^{e\varepsilon} \frac{x^l dx}{x^{r+1}} = \frac{1}{(r-l)\varepsilon^{r-l}}\big(1 - \frac{1}{e^{r-l}}\big) \underset{\varepsilon\to 0}{\longrightarrow} \infty.$$

By the same arguments as above it is easily shown that for $f \in \mathcal{G}$ we have

$$\int Q_{r-1}^{-\tau_{r-1}^0(\varepsilon)} Q_{r-2}^{-\tau_{r-2}^0(\varepsilon)} \dots Q_1^{-\tau_1^0(\varepsilon)} f dP_{g,\varepsilon} = \sum_{k=0}^\infty \frac{1}{k!} \int_{(G_\varepsilon^0)^k} \Delta^{\otimes k}(r-1) f_k d\Pi^k. \quad (55)$$

Denote the operator $Q_{r-1}^{-\tau_{r-1}^0(\varepsilon)} Q_{r-2}^{-\tau_{r-2}^0(\varepsilon)} \dots Q_1^{-\tau_1^0(\varepsilon)}$ by $\mathcal{A}_\varepsilon^0$ and its conjugate operator by $\mathcal{A}_\varepsilon^{0,*}$.

Theorem 9. *Suppose that a generalized function g acts on a test function φ as $(g,\varphi) = \frac{1}{r!}\varphi^{(r)}(0)$. Then for every $f \in \mathcal{G}$ we have*

$$\lim_{\varepsilon\to 0} \int_\Omega \mathcal{A}_\varepsilon^0 f dP_{g,\varepsilon} = (\mathcal{A}_\varepsilon^{0,*} P_{g,\varepsilon}, f) = L_g f,$$

where operators $\mathcal{A}_\varepsilon^0$ are defined by (55). That means, that the generalized function L_g is a limit of generalized functions $\mathcal{A}_\varepsilon^{0,*} P_{g,\varepsilon}$.

Proof. It remains to note that by (54) the right hand side of (55) converges to $L_g f$ as $\varepsilon \to 0$. □

Acknowledgements. This paper was partly supported by DFG 436 RUS 113/823, NSh 4472.2010.1, NSh 5931.2010.1. The third author was supported by Russian Foundation for Basic Research 09-01-00515a and 11-01-90402-Ukr_f_a.

References

[1] S.Albeverio, N.Smorodina. “A Distributional Approach to Multiple Stochastic Integrals and Transformations of the Poisson Measure”, Acta Appl. Math., Vol. 94, pp. 1-19, 2006

[2] S.Albeverio, N.Smorodina. “A distributional approach to multiple stochastic integrals and transformations of the Poisson measure. II”, Acta Appl. Math., Vol. 102, pp. 319–343, 2008

[3] L.Beghin, E.Orsingher. “The distribution of the local time for “pseudo-process” and its connection with fractional diffusion equatios”, Stochastic Process. Appl., Vol. 115, pp. 1017-1040, 2005.

[4] K.Burdzy, A.Madrecki. “Ito formula for an asymptotically 4-stable process”, Ann. Appl. Probab., Vol. 6, No. 1, pp. 200-217, 1996.

[5] Yu.L.Daletskii, S.V. Fomin. “Measures and Differential Equations on Infinite Dimensional Spaces”, Nauka, Mockow, 1984; English transl., Kluwer Academic Publishers, 1992.

[6] T.Funaki. “Probabilistic construction of the solution of some higher order parabolic differential equations”, Proc. Japan Acad. Vol. A 55, pp. 176-179, 1979.

[7] B.Gaveau, P.Sainty. “A path integral formula for certain fourth-order elliptic operators”, Lett. Math. Phys., Vol. 15, No. 4, pp. 345-350, 1988.

[8] I.M. Gel’fand, G.E. Shilov, “Generalized Functions, vol. 1, Properties and Operations”, Academic Press, New York, 1964.

[9] K.Hochberg. “A signed measure on path space related to Wiener measure”, Ann. Probab., Vol. 6, No. 3, pp. 433-458, 1978.

[10] K.Hochberg. “Central limit theorem for signed distributions”, Proc. Amer. Math. Soc., Vol. 79, No. 2, pp. 298-302, 1980.

[11] K.Hochberg, E.Orsinger. “Composition of stochastic processes governed by higher-order parabolic and hyperbolic equations”, J. Theoret Probab., Vol. 9, No. 2, pp. 511-532, 1996.

[12] J.Kerstan, K.Mattes, and J.Mecke. “Infinite divisible point processes”, Akademie-Verlag, Berlin, 1978.

[13] V.Yu. Krylov. “Some properties of the distributions corresponding to the equation $\partial u/\partial t = (-1)^{q+1}\partial^{2q}u/\partial x^{2q}$”, Dokl. Akad. Nauk SSSR, Vol. 132, pp. 1254-1257, 1996 (Russian)

[14] Y.Nikitin, E.Orsingher. “On Sojourn Distribution of Process Related to Some Higher-Order Heat-Type Equation”, Journal of Theoretical Probab., Vol.13, No 4, pp. 997-1012, 2000.

[15] A.V.Skorokhod. “Stochastic processes with independent increments”, "Nauka", Moskow, 1986; English transl., Kluver, Dordrecht, 1991.

[16] N.Smorodina, M.Faddeev. “The Lévy-Khinchin representation of the one class of signed stable measures and some its applications”, Acta Appl. Math., Vol. 110, pp. 1289-1308, 2010.